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Analytic Localization of Group Representations

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INTRODUCTION

In the last several years the technique of localization combined with the theory of $\mathscr{D}$-modules has become an important tool in studying representations of semisimple Lie algebras. Probably the most spectacular success of this approach is the proof of the Kazhdan–Lusztig conjectures. Powerful as it is, however, the existing localization theory is best suited to study primarily algebraic aspects of representation theory. For example, given a global representation of a semisimple Lie group, the localization technique is used to investigate the underlying Harish-Chandra module.

The purpose of this paper is to develop another localization technique, which we call *analytic localization*, designed to study the global representation itself. Roughly speaking, the analytic localization functor establishes an equivalence between group representations on the one hand, and certain $\mathscr{D}$-modules on the other. In particular it provides a comprehensive geometric construction of representation of semisimple Lie groups. A precise statement of our main results appears later in the introduction.

Here and throughout this paper G_0 denotes a connected semisimple Lie group with finite center. For the purpose of this introduction we assume that G_0 arises as a set of real points of a connected simply connected complex group G. We fix a maximal compact subgroup K_0 of G_0, and denote by K its complexification. We let $\mathfrak{g}$ denote the complexified Lie algebra of G_0 and by X we denote the complex manifold of all Borel subalgebras of $\mathfrak{g}$—the flag variety of $\mathfrak{g}$.

Geometric realization of representations of G_0 has been at the center of attention of people in the field. A classical result in this direction is the theorem of Borel–Weil which describes such a realization for a finite

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dimensional irreducible representation F of G_0 (and thus G): F is uniquely characterized by its lowest weight μ in the dual of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. On the other hand μ determines a G-homogeneous line bundle $\mathfrak{L}_\mu$ on the flag variety X. The associated sheaf $\mathcal{O}_\mu = \mathcal{O}_\mu(\mathfrak{L}_\mu)$ of holomorphic local sections is acyclic for the functor Γ of global sections, and F is naturally isomorphic to $\Gamma(X, \mathcal{O}_\mu)$. The same statement remains true in the algebraic context: X has an underlying structure of algebraic variety, and $\mathfrak{L}_\mu$ is an algebraic line bundle on X. One simply replaces $\mathcal{O}_\mu$ by the sheaf $\mathcal{O}_\mu^{\mathrm{alg}}$ of regular sections of $\mathfrak{L}_\mu$. Let $\mathfrak{b}_x$ be the Borel subalgebra parametrized by $x \in X$, and $\mathfrak{n}_x$ its nilpotent radical. It is important to realize that the fiber of $\mathfrak{L}_\mu$ (equivalently, the geometric fiber of $\mathcal{O}_\mu$, or of $\mathcal{O}_\mu^{\mathrm{alg}}$) is naturally isomorphic as a $\mathfrak{b}_x$-module to the zero $\mathfrak{n}_x$-homology group $F/\mathfrak{n}_x F$ of F.

The correspondence Δ^{alg}: $F \to \mathcal{O}_\mu^{\mathrm{alg}}$ is the simplest, but a very important, example of the classical localization functor, which we will refer to from now on as the algebraic localization functor. The analytic localization, which is developed in this paper, assigns to F the sheaf $\mathcal{O}_\mu$. Aside from the difference between "algebraic" and "analytic" (which in this case is inessential) these two functors are the same in this case. As we shall see later, this is the ony case in which they agree, although strong formal similarities remain.

Geometric constructions of infinite dimensional representations of G_0 have encountered a number of serious technical difficulties. For a long time the only general geometric model available was that of a principal series (a representation induced from a finite dimensional representation of a parabolic subgroup). The very nontrivial case of discrete series was first solved by Schmid in his thesis [32]: they are represented as Dolbeault cohomology groups, in a specific degree, of a G_0-homogeneous line bundle on an open G_0-orbit on X. In every other degree the cohomology of this line bundle is zero. Schmid's result illustrates several difficulties. The first is a lack of general vanishing theorems for vector bundles on G_0-homogeneous manifolds. The vanishing theorem in [32] was proved by ad hoc methods, and only for "sufficiently nonsingular" parameters (this restriction was later removed using the technique of tensoring with finite dimensional representations [42]). Then, it is not obvious at all that in general the nonvanishing cohomology will have a useful Hausdorff topology. In the case of [32] this is done by identifying the cohomology group in question with a solution space of a certain elliptic differential operator on G_0/K_0. The resulting topology turns out to be Fréchet. To exhibit the discrete series as unitary representations one has to relate the Dolbeault and L^2-cohomologies. This brings us to the next difficulty one needs to face developing a general geometric realization model: several topologically nonequivalent representations may have isomorphic Harish-Chandra modules of their K_0-finite vectors.

The algebraic localization theory, which was developed by Beilinson and Bernstein [1] and, in a more specific setting, by Brylinski and Kashiwara [8], disposes of the topological difficulties by dealing with Harish-Chandra modules, and, more generally, with $\mathfrak{g}$-modules with a given regular infinitesimal character. It captures the beauty and simplicity of the Borel–Weil theorem. Below we sketch the essential features of this theory.

For an integral $\lambda \in \mathfrak{h}^*$ denote by $\mathscr{D}_\lambda^{\mathrm{alg}}$ the sheaf of algebraic differential operators on $\mathcal{O}_{\lambda+\rho}^{\mathrm{alg}}$. One observes that $\mathscr{D}_\lambda^{\mathrm{alg}}$ is generated by the structure sheaf $\mathcal{O}^{\mathrm{alg}}$ of X and the action of $\mathfrak{g}$, together with a relation involving λ. The nature of this relation has nothing to do with integrality of λ. Thus, to each $\lambda \in \mathfrak{h}^*$ one can canonically attach a sheaf $\mathscr{D}_\lambda^{\mathrm{alg}}$ of algebras on X. This is called the sheaf of twisted (by λ) algebraic differential operators on X. It reduces to the ordinary sheaf $\mathscr{D}^{\mathrm{alg}}$ of algebraic differential operators on X if $\lambda = -\rho$. In general, the statement remains true locally.

On the other hand, $\lambda \in \mathfrak{h}^*$ (or rather its Weyl group orbit) parametrizes a character ϕ of the center of the universal enveloping algebra of $\mathfrak{g}$. Denote by U_λ the quotient of $U(\mathfrak{g})$ by the ideal generated by the kernel of ϕ. Clearly U_λ depends only on the Weyl group orbit of λ but it is convenient to keep the dependence on λ explicit. We note that a U_λ-module is nothing but a $\mathfrak{g}$-module with infinitesimal character ϕ. It turns out that $\mathscr{D}_\lambda^{\mathrm{alg}}$ is an acyclic sheaf with algebra of global sections naturally isomorphic to U_λ. This important fact allows us to define the algebraic localization functor

$$\Delta^{\mathrm{alg}}(V) = \mathscr{D}_\lambda^{\mathrm{alg}} \otimes_{U_\lambda} V$$

from the category of U_λ-modules to the category of (quasicoherent) $\mathscr{D}_\lambda^{\mathrm{alg}}$-modules. The functor of global sections, going in the opposite direction, is the right adjoint to $\mathscr{D}_\lambda^{\mathrm{alg}}$. The main theorem of Beilinson and Bernstein [1] asserts that, for λ regular and antidominant, Δ^{alg} is an equivalence of categories, with the functor Γ of global sections serving as its inverse. Moreover, the higher derived functors of Δ^{alg} and Γ vanish. This is a vast generalization of the Borel–Weil theorem: in that special case $\Delta^{\mathrm{alg}}(F) = \mathcal{O}_\mu^{\mathrm{alg}}$. When restricted to the category of Harish-Chandra U_λ-modules, Δ^{alg} establishes an equivalence of this category with the category of K-equivariant coherent $\mathscr{D}_\lambda^{\mathrm{alg}}$-modules on X. Since K acts on X algebraically with finitely many orbits, the latter category consists of holonomic modules with regular singularities. This allows for a very straightforward classification of irreducible objects, in terms of K-equivariant connections on orbits.

In order to develop a workable localization theory for group representations, one needs to have a canonical way of extending a Harish-Chandra module to a representation of G_0. Several such "globalization functors" exist by now [40, 33]. The one which is most useful for us is Schmid's functor of minimal globalization [33]. The minimal globalization of a Harish-Chandra module for G_0 is the minimal topological vector space completion of the module to a module on which G_0 acts. It is isomorphic to the space of analytic vectors in any Banach space globalization of the module in question (cf. [33]). Our intent here is to develop a localization theory which will also represent the minimal globalizations of Harish-Chandra modules in terms of a tractable category of sheaves on X. The algebraic localization, which is defined in terms of tensor product, will not do here because it does not take into account the topology of a module that happens to be a topological module. Thus, one would expect the algebraic localization of the minimal globalization of a module to be extremely complicated in comparison to the localization of the original and not recognizable as arising from its completion. Our analytic localization, on the other hand, represents minimal globalizations of Harish-Chandra modules in terms of sheaves of $\mathscr{D}_\lambda$-modules ($\mathscr{D}_\lambda$ is a sheaf of twisted differential operators with holomorphic coefficients) which have a particularly simple form: they are sheaves whose restrictions to any G_0-orbit in X are locally free of finite rank as modules over the sheaf of germs of holomorphic functions on X. We should point out that, except for trivial cases, these sheaves are not coherent. On the other hand, however, they bear striking resemblance to constructible sheaves. In fact, there is an equivalence between these sheaves and constructible sheaves which is quite simple and transparent.

Denote by $\mathfrak{M}(U_\lambda, G_0)$ the category of minimal globalizations of Harish-Chandra U_λ-modules. The modules in this category belong to a particularly good class of topological vector spaces—duals of nuclear Fréchet spaces or DNF spaces. This class of spaces behaves well under completed topological tensor product, which is fortunate since our version of localization makes heavy use of this operation.

The construction of $\mathscr{D}_\lambda^{\mathrm{alg}}$ can be carried out in the analytic context. This results in a sheaf $\mathscr{D}_\lambda$ of twisted differential operators with holomorphic coefficients. This also is an acyclic sheaf with algebra of global sections isomorphic to U_λ. The sheaf $\mathscr{D}_\lambda$ has the structure of a DNF-sheaf—that is, its sections over any compact subset of K form a DNF-space. In particular, U_λ is a DNF-space: it is a countable direct limit of its finite dimensional subspaces.

Denote by $\mathfrak{M}(U_\lambda)$ the category of DNF U_λ-modules, and, for an object V in $\mathfrak{M}(U_\lambda)$, set

$$\Delta(V) = \mathscr{D}_\lambda \mathbin{\hat{\otimes}}_{U_\lambda} V = \mathrm{Tor}_0^{U_\lambda}(\mathscr{D}_\lambda, V),$$

where $\hat{\otimes}$ denotes the completed topological tensor product. The functor into the category of sheaves defined in this way *need not be exact*, though it is right exact. It turns out, however, that the category $\mathfrak{M}(U_\lambda)$ has enough topologically projective resolutions to allow us to do homological algebra and define derived functors $D_k\Delta$ for Δ [37].

The failure of Δ to be exact forces us to work in derived (localized) categories. Let $D\mathfrak{M}(U_\lambda)$ denote the category obtained by localization of the homotopy category of $\mathfrak{M}(U_\lambda)$ with respect to quasi-isomorphisms. The category $D\mathfrak{M}(\mathscr{D}_\lambda)$ is constructed in a similar manner from the category $\mathfrak{M}(\mathscr{D}_\lambda)$ of DNF $\mathscr{D}_\lambda$-modules. One defines a derived functor $D\Delta$: $D\mathfrak{M}(\mathscr{D}_\lambda) \to D\mathfrak{M}(\mathscr{D}_\lambda)$ of Δ—this is the *functor of analytic localization*—and a derived functor $D\Gamma$: $D\mathfrak{M}(\mathscr{D}_\lambda) \to D\mathfrak{M}(U_\lambda)$ of the functor of global sections. The first main result of this paper is the assertion (cf. Theorem 5.4):

> If λ is regular, then $D\Delta$: $D\mathfrak{M}(U_\lambda) \to D\mathfrak{M}(\mathscr{D}_\lambda)$ is an equivalence of categories with inverse functor $D\Gamma$.

Unlike the case of algebraic localization, even for λ antidominant one does not have an equivalence without passing to derived categories.

The equivalence between $D\mathfrak{M}(U_\lambda)$ and $D\mathfrak{M}(\mathscr{D}_\lambda)$ is largely of technical interest, since these categories are too large to be of great interest in themselves. However, in the second half of the paper we establish our main result: If this equivalence is restricted to the full subcategory of $D\mathfrak{M}(U_\lambda)$ generated (in the sense of triangulated categories) by minimal globalizations of Harish-Chandra modules, then it determines an equivalence between this category and a particularly simple category of $\mathscr{D}_\lambda$-modules.

More specifically, let $D\mathfrak{M}(U_\lambda, G_0)$ be the full triangulated subcategory of $D\mathfrak{M}(U_\lambda)$ generated by minimal globalizations of Harish-Chandra modules. Similarly, let $D\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ be the full triangulated subcategory of $D\mathfrak{M}(\mathscr{D}_\lambda)$ generated by the category of G_0-equivariant $\mathscr{D}_\lambda$-modules whose restrictions to any G_0-orbit are locally free of finite rank over the sheaf of germs of holomorphic functions on X. Our main theorem (Theorem 10.9) asserts that:

> The functor $D\Delta$ determines an equivalence of categories between $D\mathfrak{M}(U_\lambda, G_0)$ and $D\mathfrak{M}(\mathscr{D}_\lambda, G_0)$.

To understand this result, it will be helpful to look at a simple example: the case where $G_0 = SU(1, 1)$, and $K_0 =$ diagonal matrices in $SU(1, 1)$.

In this case $G = SL(2, \mathbb{C})$, K is its subgroup of diagonal matrices, and $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. The flag variety $X = \mathbb{C} \cup \{\infty\}$. The group G acts transitively on X via linear fractional transformations. The orbits of G_0 are the open unit disc $S^+ = \{z: |z| < 1\}$, the unit circle $S^0 = \{z: |z| = 1\}$, and S^-—the exterior of the closed unit disc, including ∞. The K-orbits are $Q^+ = \{0\}$, $Q^0 = \mathbb{C} - \{0\}$, and $Q^- = \{\infty\}$. The G_0-orbits are associated to K-orbits in the order indicated. This is a special case of the Matsuki correspondence.

Consider the case of $\lambda = -\rho$. Let F be the trivial Harish-Chandra module, and let I be the principal series Harish-Chandra module of length three containing F as a sub, and the holomorphic and antiholomorphic discrete series V_+ and V_- as quotients. We have an exact sequence

$$0 \to F \to I \to V_+ \oplus V_- \to 0.$$

Applying Δ^{alg} we get an exact sequence

$$0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{V}_+ \oplus \mathscr{V}_- \to 0$$

of $\mathscr{D}^{\mathrm{alg}}$-modules, where $\mathscr{F} = \mathcal{O}^{\mathrm{alg}}$ and $\mathscr{I}$ is the sheaf of meromorphic functions on X with poles at 0 and ∞ only. The quotient sheaf $\mathscr{I}/\mathscr{F}$ decomposes as a direct sum of two skyscraper sheaves $\mathscr{V}_+$ and $\mathscr{V}_-$ supported in $\{0\}$ and $\{\infty\}$, respectively. By passing to global sections we recover the preceding exact sequence of Harish-Chandra modules. This simple example illustrates the Beilinson–Bernstein equivalence theorem.

Since the functor $(\ )^{\sim}$ of minimal globalization is exact [33], we have an exact sequence

$$0 \to F^{\sim} \to I^{\sim} \to V_+^{\sim} \oplus V_-^{\sim} \to 0.$$

The minimal globalization of F is of course F itself. Since F is obtained as global sections of an acyclic DNF $\mathscr{D}$-module $\mathcal{O}$, the equivalence of categories result implies that $D\Delta(F) = \mathcal{O}$ (we regard U_λ-modules and $\mathscr{D}_\lambda$-sheaves as objects in the corresponding derived categories by identifying them with complexes concentrated in degree zero only). Similarly, $I^{\sim}$ can be realized as the space of global sections of the DNF $\mathscr{D}$-module $\mathscr{J}$, obtained by first restricting $\mathcal{O}$ to S^0, and then extending by zero to X. Since $\mathscr{J}$ is acyclic it follows that $D\Delta(I^{\sim}) = \mathscr{J}$. Now, we have an exact sequence of sheaves $0 \to \mathscr{W}_+ \oplus \mathscr{W}_- \to \mathcal{O} \to \mathscr{J} \to 0$, where $\mathscr{W}_\pm$ is obtained by extending the sheaf of holomorphic functions on $S^{\pm}$ by zero to X. This exact sequence determines a distinguished triangle in the derived category. By turning we obtain a triangle

$$\mathcal{O} \to \mathscr{J} \to \mathscr{W}_+[1] \oplus \mathscr{W}_-[1],$$

where "[1]" is the translation operator. Since $D\Delta$ preserves triangles, it follows that $D\Delta V_+^{\sim} = \mathscr{W}_+[1]$, and $D\Delta V_-^{\sim} = \mathscr{W}_-[1]$. One recovers $V_\pm^{\sim}$ as the zero hypercohomology of $\mathscr{W}_\pm[1]$ or, equivalently, as the first cohomology of $\mathscr{W}_\pm$.

Restricted to $\mathfrak{M}(U_\lambda, G_0)$, $D\Delta$ is an equivalence between minimal globalizations and "perverse sheaves," which are the objects in

$D\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ with hypercohomology concentrated in degree zero. In general such an object can be very complicated. One explanation of this stems from the relationship between derived functors of localization and $\mathfrak{n}_x$-homology.

It follows rather easily from the definitions that, for a regular λ, the geometric fiber of $\Delta(V)$ at x is isomorphic as a $\mathfrak{b}_x$-module to the $(\lambda+\rho)$-weightspace of $H_0(\mathfrak{n}_x, V)$. The proof is identical to that of an analogous result for Δ^{alg} [2]. If V is a minimal globalization, the stalk of $D_k\,\Delta(V)$ at x is a free $\mathscr{O}_x$-module of finite rank. It follows that the geometric fiber of $D_k\,\Delta(V)$ at x is isomorphic to the $(\lambda+\rho)$-weightspace of $H_k(\mathfrak{n}_x, V)$. As an application we get:

> The homology groups of a minimal globalization of a Harish-Chandra module with respect to a maximal nilpotent subalgebra of $\mathfrak{g}$ are finite dimensional.

This is proved in Proposition 10.10 in the case of regular infinitesimal character. The extension to a general situation is straightforward.

Unlike in the simple example of $SU(1, 1)$ we discussed above, it may happen that for a minimal globalization the same weight $\lambda+\rho$ may contribute to $\mathfrak{n}$-homology in various degrees. This may serve as an indication of a rather complicated structure of the "perverse sheaves."

The organization of the paper is as follows. In Section 1 we review basic facts about the sheaf of differential operators $\mathscr{D}_\lambda$. The necessary machinery for doing homological algebra and localization in the context of topological modules over a topological algebra is discussed in Section 2. In Section 3 we consider DNF $\mathscr{D}_\lambda$-modules, which are the kinds of sheaves we hope to get from localizing DNF U_λ-modules. To facilitate the discussion we have included an appendix on topological vector spaces. We also discuss a particularly useful fine resolution for such sheaves, which is modeled on the Čech construction. The equivalence of categories between $D\mathfrak{M}(U_\lambda)$ and $D\mathfrak{M}(\mathscr{D}_\lambda)$ is proved in Section 5. The main ingredient of the proof is a vanishing theorem proved in Section 4. Here is where the theories of algebraic and analytic localization depart.

The remainder of the paper is mainly devoted to the proof of equivalence of $D\mathfrak{M}(U_\lambda, G_0)$ and $D\mathfrak{M}(\mathscr{D}_\lambda, G_0)$. The strategy of the proof relies heavily on the properties of intertwining functors. Such a functor, analogous to that introduced by Beilinson and Bernstein in [2], establishes an equivalence of $D\mathfrak{M}(\mathscr{D}_\lambda)$ and $D\mathfrak{M}(\mathscr{D}_\mu)$, for λ and μ in the same Weyl group orbit, and commutes with $D\Gamma$ and the respective functors of analytic localization at λ and μ (Sect. 6). Most of Section 9 is devoted to showing that this functor induces an equivalence of $D\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ and $D\mathfrak{M}(\mathscr{D}_\mu, G_0)$. This follows from careful $\mathfrak{sl}_2(\mathbb{C})$-type computations, partially carried out in the appendix.

It is well known that there is a one-to-one correspondence between K-orbits and G_0-orbits on X (cf. [24]). This is the Matsuki correspondence illustrated before in the case of $SU(1, 1)$. Let Q be a K-orbit and S the corresponding G_0-orbit. Also, let x be a point of $Q \cap S$. The isotropy group of x in G_0 contains a Cartan subgroup C_0, stable under the Cartan involution. We identify $\mathfrak{h}$ with the complexified Lie algebra of C_0. Let λ have antidominant real part. To each character L_λ of C_0, with differential equal to $\lambda + \rho$, one can canonically attach a "direct image" $(\mathscr{D}_\lambda^{\mathrm{alg}}, K)$-module, supported on the closure of Q (cf. [1]), and, via the global section functor, which in this case is an equivalence of categories, the standard Harish-Chandra module $I(x, L_\lambda)$ of Beilinson and Bernstein. On the other hand, this character parametrizes a "standard $\mathscr{D}_\lambda$-module" (in our sense) $\mathscr{I}(x, L_\lambda)$ supported on S. As modules of the form $\mathscr{I}(x, L_\lambda)$ and $I(x, L_\lambda)$ generate the respective triangulated categories $D\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ and $D\mathfrak{M}(U_\lambda, G_0)$, the equivalence of these categories is a direct consequence of the following proposition (cf. Proposition 10.8):

> The cohomology of $\mathscr{I}(x, L_\lambda)$ vanishes in degrees different from $c = \operatorname{codim}_{\mathbb{C}} Q$, and $H^c(X, \mathscr{I}(x, L_\lambda))$ is the minimal globalization of the standard module $I(x, L_\lambda)$.

A special case of this proposition is illustrated in the $SU(1, 1)$ example.

The above proposition gives a rather good geometric picture of how the localization of a Harish-Chandra module is related to the analytic localization of its minimal globalization, at least in the case of standard modules. One application of this is the following comparison theorem: if $\mathfrak{n}$ is the nilpotent radical of a Borel subalgebra corresponding to a point, such as x, which is in the intersection of a K-orbit with its associated G_0-orbit, then the $\mathfrak{n}$-homology of any Harish-Chandra module is the same as the $\mathfrak{n}$-homology of its minimal globalization. The proof of this result will appear in a forthcoming paper.

Proposition 10.8 is related to a result of Schmid and Wolf on maximal globalizations of Vogan–Zuckerman standard modules [34]. To a datum (x, C_0, L) one can attach a series of standard Vogan–Zuckerman modules, which are related to an appropriate Beilinson–Bernstein standard module by duality. It turns out the maximal globalization of such modules can be realized geometrically in a number of ways, for example in terms of local cohomology along S of the standard G_0-equivariant sheaf attached to L [34]. In principle there is a duality between our model of minimal globalization of standard modules and the maximal globalization defined in [34]. It is not clear, however, how to carry this out geometrically.

Both the proofs of Proposition 10.8 and the result in [34] employ an idea which has originated in [20]: There is another datum (S', x', L_μ) with

μ not necessarily antidominant, for which the assertion follows (in this case from results in Schmid's thesis [32]). Then one relates the objects associated to the data (S, x, L_λ) and (S', x', L_μ) by some version of an intertwining functor. In our case it suffices to show that the intertwining functor relating $D\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ and $D\mathfrak{M}(\mathscr{D}_\mu, G_0)$ maps $\mathscr{I}(x, L_\lambda)$ into $\mathscr{I}(x', L_\mu)$. This reduces to an $\mathfrak{sl}_2(\mathbb{C})$ calculation carried out in the appendix.

Sections 7 and 8 are technical in nature. They are devoted to discussion of analytic modules and sheaves equipped with a group action, and their behavior under localization and cohomology.

The results of this paper were announced, in basic outline, at the Autumn 1986 AMS regional meeting in Logan, Utah.

1. Sheaves of Differential Operators

The purpose of this section is to review the essential facts concerning standard sheaves of differential operators on the flag manifold.

Here and throughout the paper G will be a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$ and X will be the corresponding flag manifold, i.e., the manifold of all Borel subalgebras of $\mathfrak{g}$. Thus, X is a complex analytic G-homogeneous space and, for each $x \in X$, the isotropy group of x is a Borel subgroup B_x. The unipotent radical of B_x will be denoted N_x. The quotient $H_x = B_x/N_x$ is isomorphic to a Cartan subgroup of G. We denote the Lie algebras of B_x, N_x, and H_x by $\mathfrak{b}_x$, $\mathfrak{n}_x$, and $\mathfrak{h}_x$. Each of the collections $\{\mathfrak{b}_x\}$, $\{\mathfrak{n}_x\}$, and $\{\mathfrak{h}_x\}$ forms a holomorphic G-homogeneous bundle of Lie algebras over X. We denote these bundles by $\mathfrak{B}$, $\mathfrak{N}$, $\mathfrak{H}$, and their sheaves of sections by $\mathscr{B}$, $\mathscr{N}$, and $\mathscr{H}$. The adjoint representation of B_x on $\mathfrak{h}_x$ is trivial, which implies $\mathscr{H} = \mathcal{O} \otimes \mathfrak{h}$, where $\mathfrak{h}$ is the space of constant sections and $\mathcal{O}$ is the sheaf of germs of holomorphic functions. We call $\mathfrak{h}$ *the abstract Cartan subalgebra* of $\mathfrak{g}$. This terminology is justified by the fact that, for each $x \in X$, $\mathfrak{h}$ is isomorphic to $\mathfrak{h}_x$, via the "evaluation at x" map.

There is a natural sheaf of algebras $\mathscr{D}_\mathfrak{h}$ associated with this picture as follows (cf. [2, 19, 25, 26], etc): The sheaf $\mathcal{O} \otimes U(\mathfrak{g})$, i.e., the sheaf of germs of $U(\mathfrak{g})$-valued finite rank holomorphic functions on X, is in a natural way a $(\mathcal{O}, U(\mathfrak{g}))$-bimodule. There is a unique algebra structure on $\mathcal{O} \otimes U(\mathfrak{g})$ given by specifying that $\mathcal{O} \otimes 1$ and $1 \otimes U(\mathfrak{g})$ be subalgebras with the standard operations, and that the commutation relations $[1 \otimes \xi, f \otimes 1] = \xi(f) \otimes 1$ be satisfied for $\xi \in \mathfrak{g}$ and $f \in \mathcal{O}$, where ξ acts on f through the differential of the action of G on X: $\xi(f)(x) = d/dt|_{t=0} f(\exp(-t\xi)x)$. We denote this algebra by $U(\mathscr{G})$. The motivation for this notation is the following. Let $\mathscr{G} = \mathcal{O} \otimes \mathfrak{g}$. This trivial $\mathcal{O}$-module inherits from $U(\mathscr{G})$ the structure of a sheaf of Lie algebras, and $U(\mathscr{G})$ has the formal properties of its universal enveloping algebra. A calculation shows that the right ideal in $U(\mathscr{G})$ generated by $\mathscr{N}$ is actually a two-sided ideal, and we define $\mathscr{D}_{\mathfrak{h}}$ to be the associated quotient sheaf. An alternate description which is sometimes useful is the following:

If we choose a base point x_0 of X then we determine a projection π: $G \to X$ by $\pi(g) = gx_0$; in other words, we regard X as G/B_{x_0}. The subgroup N_{x_0} determines a G-homogeneous analytic fiber bundle ρ: $G/N_{x_0} \to G/B_{x_0}$ with fiber $H_{x_0} = B_{x_0}/N_{x_0}$. In fact, it is a principal H_{x_0}-bundle under the right action of H_{x_0} on G/N_{x_0}. In this setting, the sheaves $U(\mathscr{G})$ and $\mathscr{D}_{\mathfrak{h}}$ may be described as follows:

1.1. PROPOSITION. *Given a base point x_0 for X, the sheaf $U(\mathscr{G})$ is naturally isomorphic to the direct image under π of the sheaf of right B_{x_0}-invariant differential operators on G, while the sheaf $\mathscr{D}_{\mathfrak{h}}$ is naturally isomorphic to the direct image under ρ of the sheaf of holomorphic differential operators on G/N_{x_0} which are right H_{x_0}-invariant.*

Proof. If V is a domain in G/B_{x_0}, we let $E(V)$ denote the algebra of all holomorphic differential operators on $\pi^{-1}(V)$ which are right invariant with respect to B_{x_0}. If we think of the enveloping algebra $U(\mathfrak{g})$ as the algebra of holomorphic differential operators on G which are right G-invariant then $U(\mathfrak{g})$ is naturally a subalgebra of $E(V)$. The algebra $\mathcal{O}(V) = \Gamma(V, \mathcal{O})$ may also be regarded as the subalgebra of degree zero operators in $E(V)$ via the map $f \to f \circ \pi$. Thus, there is a map $f \otimes v \to (f \circ \pi)v$: $\mathcal{O}(V) \otimes U(\mathfrak{g}) \to E(V)$. On passing to germs, this defines a map from $U(\mathscr{G})$ to the direct image under π of the sheaf of right B_{x_0}-invariant differential operators on G. It is easy to see that this is, in fact, an algebra isomorphism if $U(\mathscr{G})$ is given the multiplication defined earlier. This isomorphism takes $\mathscr{N}$ to the direct image under π of the sheaf of germs of holomorphic vector fields on G which are tangent to the left cosets of N_{x_0}. These vector fields form an ideal in the Lie algebra of right B_{x_0}-invariant holomorphic vector fields. The right ideal they generate in the sheaf of right B_{x_0}-invariant differential operators is, thus, a two-sided ideal and, in fact, consists precisely of operators which annihilate functions constant on left cosets of N_{x_0}. It follows that $\mathscr{D}_{\mathfrak{h}}$ is naturally isomorphic to the direct image under ρ of the sheaf of holomorphic differential operators on G/N_{x_0} which are right H_{x_0}-invariant.

Since $\mathscr{D}_{\mathfrak{h}}$ is obtained by dividing $U(\mathscr{G})$ by the ideal generated by $\mathscr{N}$, the space $\mathfrak{h}$ of global sections of $\mathscr{H}$ maps into $\Gamma(\mathscr{D}_{\mathfrak{h}})$. It follows that there is a natural homomorphism $U(\mathfrak{h}) \to \Gamma(\mathscr{D}_{\mathfrak{h}})$. The embedding $\xi \to 1 \otimes \xi$: $U(\mathfrak{g}) \to \Gamma(U(\mathscr{G}))$ results in a homomorphism of $U(\mathfrak{g})$ into $\Gamma(\mathscr{D}_{\mathfrak{h}})$. If we use the picture of $\mathscr{D}_{\mathfrak{h}}$ elucidated in Proposition 1.1, then $\Gamma(\mathscr{D}_{\mathfrak{h}})$ is represented as the algebra of holomorphic differential operators on G/N_{x_0} which commute with the right action of H_{x_0}. The above natural image of $U(\mathfrak{h})$ in $\Gamma(\mathscr{D}_{\mathfrak{h}})$ appears as the algebra of differential operators generated by the infinitesimal generators of the right H_{x_0}-action. Clearly this algebra lies in the center of $\Gamma(\mathscr{D}_{\mathfrak{h}})$. In particular, its elements commute with the elements of the natural image of $U(\mathfrak{g})$ in $\Gamma(\mathscr{D}_{\mathfrak{h}})$, from which we conclude that there is a natural algebra homomorphism $U(\mathfrak{h}) \otimes U(\mathfrak{g}) \to \Gamma(\mathscr{D}_{\mathfrak{h}})$. Each element z of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ agrees mod the right ideal generated by $\mathfrak{n}_x$ with a unique element $\gamma_x(z)$ of $U(\mathfrak{h}_x)$. The collection $\{\gamma_x(z)\}_{x \in X}$ determines an element $\gamma(z) \in U(\mathfrak{h})$. In this way we obtain the (unnormalized) Harish-Chandra homomorphism γ: $Z(\mathfrak{g}) \to U(\mathfrak{h})$ (cf. [16, Part III; 13, Chap. 7, Sect. 4]) which allows us to regard $U(\mathfrak{h})$ as a $Z(\mathfrak{g})$-module. It is immediate from its construction that the homomorphism $U(\mathfrak{h}) \otimes U(\mathfrak{g}) \to \Gamma(\mathscr{D}_{\mathfrak{h}})$ induces a homomorphism $U(\mathfrak{h}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{g}) \to \Gamma(\mathscr{D}_{\mathfrak{h}})$.

The following proposition is well known. The proof given in [26] is particularly simple.

1.2. PROPOSITION. *The sheaf $\mathscr{D}_{\mathfrak{h}}$ is acyclic and the natural algebra homomorphism $U(\mathfrak{h}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{g}) \to \Gamma(\mathscr{D}_{\mathfrak{h}})$ is an isomorphism.*

One of the main ideas of the proof of Proposition 1.2 given in [26] involves a special resolution of $\mathscr{D}_{\mathfrak{h}}$, which will be of fundamental importance to us. First, we need a brief discussion of the *Koszul Complex*, $K(\mathfrak{n}, M)$, for a Lie algebra $\mathfrak{n}$ and an $\mathfrak{n}$-module M. This is the complex

$$0 \to (\textstyle\bigwedge^n \mathfrak{n}) \otimes M \to \cdots \to (\textstyle\bigwedge^2 \mathfrak{n}) \otimes M \to \mathfrak{n} \otimes M \to M \to 0,$$

where $\bigwedge^p \mathfrak{n}$ is the p-fold exterior product of $\mathfrak{n}$ and n is the dimension of $\mathfrak{n}$. The homology of this complex is the Lie algebra homology of $\mathfrak{n}$ with coefficients in M and is sometimes called the *$\mathfrak{n}$-homology of M*; it is also $\mathrm{Tor}^{U(\mathfrak{n})}(\mathbb{C}_0, M)$, where $\mathbb{C}_0$ is the trivial one-dimensional $\mathfrak{n}$-module. For information on this complex and its role in homology see [9, Chap. XIII, Sect. 7; 21, also 11]. The degree zero map $\mathfrak{n} \otimes M \to M$ is just the module action map $\eta \otimes m \to \eta m$ and so $H_0(K(\mathfrak{n}, M)) \cong M/\mathfrak{n}M$. The homology of the Koszul complex vanishes except in degree zero if M is projective as a $U(\mathfrak{n})$-module.

Let us return to the flag manifold X for G. Let M be a $\mathfrak{g}$-module and consider the Koszul complex $K(\mathfrak{n}_x, M)$. If we vary x over X, we obtain a complex of vector bundles and bundle maps

$$0 \to (\textstyle\bigwedge^n \mathfrak{N}) \otimes M \to \cdots \to (\textstyle\bigwedge^2 \mathfrak{N}) \otimes M \to \mathfrak{N} \otimes M \to X \times M \to 0,$$

which we shall denote $K(\mathfrak{N}, M)$. The corresponding complex of sheaves of local sections will be denoted $K(\mathscr{N}, M)$. As we shall see, this complex of sheaves is particularly interesting when $M = U(\mathfrak{g})$, with $U(\mathfrak{g})$ considered as a left $\mathfrak{g}$-module. In fact, $K(\mathcal{N}, U(\mathfrak{g}))$ is a complex of the form

$$0 \to (\textstyle\bigwedge^n \mathcal{N}) \otimes_{\mathcal{O}} U(\mathcal{G}) \to \cdots \to (\textstyle\bigwedge^2 \mathcal{N}) \otimes_{\mathcal{O}} U(\mathcal{G})$$
$$\to \mathcal{N} \otimes_{\mathcal{O}} U(\mathcal{G}) \to U(\mathcal{G}) \to 0, \tag{1.1}$$

where $\mathcal{N} \otimes_{\mathcal{O}} U(\mathcal{G}) \to U(\mathcal{G})$ is given by multiplication in $U(\mathcal{G})$. By definition, $\mathcal{D}_{\mathfrak{h}}$ is the cokernel of this map. The complex (1.1) is a complex of sheaves of $U(\mathcal{G})$-bimodules, where in each case the right $U(\mathcal{G})$-action is given by right multiplication on the $U(\mathcal{G})$ factor, while the left action on $(\bigwedge^p \mathcal{N}) \otimes_{\mathcal{O}} U(\mathcal{G})$ is determined by letting $\mathfrak{g}$ act through the tensor product of the left actions of $\mathfrak{g}$ on $U(\mathcal{G})$ and $\bigwedge^p \mathcal{N}$. Note that each $(\bigwedge^p \mathcal{N}) \otimes_{\mathcal{O}} U(\mathcal{G})$ is locally free as a right $U(\mathcal{G})$-module, since $\bigwedge^p \mathcal{N}$ is a locally free $\mathcal{O}$-module.

1.3. PROPOSITION. *The Koszul complex $K(\mathcal{N}, U(\mathfrak{g}))$ provides a locally free right $U(\mathcal{G})$-module resolution of $\mathcal{D}_{\mathfrak{h}}$, which is locally split as a complex of sheaves of topological vector spaces.*

Proof. By the Poincaré–Birkoff–Witt Theorem $U(\mathfrak{g})$ is a free left $U(\mathfrak{n}_x)$-module and, hence, $H_p(K(\mathfrak{n}_x, U(\mathfrak{g})))$ vanishes, except in degree zero where it is $U(\mathfrak{g})/\mathfrak{n}_x U(\mathfrak{g})$. It is not immediately clear that this implies that the complex of sheaves of local sections, $K(\mathcal{N}, U(\mathfrak{g}))$, is also exact except in degree zero. Nevertheless, it is true, because the complex $K(\mathfrak{N}, U(\mathfrak{g}))$ of bundles and bundle maps is locally trivial. That is, if we choose a point x_0 and a local section σ of the map $g \to gx_0 \colon G \to X$, then conjugation by $\sigma(x)^{-1}$ transforms $K(\mathfrak{N}, U(\mathfrak{g}))$ into the trivial bundle with fiber $K(\mathfrak{n}_{x_0}, U(\mathfrak{g}))$ for x near x_0. In fact, if we choose a contracting homotopy for the augmented complex $K(\mathfrak{n}_{x_0}, U(\mathfrak{g})) \to U(\mathfrak{g})/\mathfrak{n}_{x_0} U(\mathfrak{g}) \to 0$ as a complex of vector spaces and then conjugate it by $\sigma(x)$ for x near x_0, we construct a contracting homotopy for the corresponding complex of bundles in a neighborhood of x. This proves not only the exactness of the associated complex of sheaves of local sections but also the fact that it is locally split as a complex of sheaves of topological vector spaces.

Recall that the abstract Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is the space of sections of the trivial G-equivariant sheaf $\mathcal{B}/\mathcal{N}$ on X. The sheaf $\mathcal{B}$ of Lie algebras acts on $\mathcal{G} = \mathcal{O} \otimes \mathfrak{g}$ by the ad representation. Consider a maximal increasing filtration of $\mathcal{G}$ by $\mathcal{B}$-invariant G-equivariant $\mathcal{O}$-submodules. Then $\mathcal{N}$ acts on the associated graded module $\operatorname{Gr} \mathcal{G}$ trivially, and, as a result, $\operatorname{Gr} \mathcal{G}$ becomes a semisimple $\mathfrak{h}$-module. The nonzero weights $\Delta \subset \mathfrak{h}^*$ of the action of $\mathfrak{h}$ on $\operatorname{Gr} \mathcal{G}$ form a root system, which we call the *abstract root system of* $(\mathfrak{g}, \mathfrak{h})$. The weights of $\operatorname{Gr} \mathcal{N} \subset \operatorname{Gr} \mathcal{G}$ cut out a system of positive roots Δ^+

inside Δ. We denote by W the Weyl group of Δ, i.e., the group generated by reflections $s(\alpha)$, for $\alpha \in \Delta$. We want to relate these abstract data to the "usual" ones. For each $x \in X$ consider the natural isomorphism $\mathfrak{h} \to \mathfrak{h}_x = \mathfrak{b}_x/\mathfrak{n}_x$ which assigns to each section in $\mathfrak{h}$ its value at x. The inverse of the dual of this map is an isomorphism $\lambda \to \lambda_x$: $\mathfrak{h}^* \to \mathfrak{h}_x^*$. We refer to both of these isomorphisms as *specialization at* x. Specialization at x identities Δ, Δ^+ with subsets Δ_x, Δ_x^+ of $\mathfrak{h}_x^*$. If we choose a Cartan subalgebra $\mathfrak{c}$ in $\mathfrak{b}_x$, and naturally identify $\mathfrak{c}$ with $\mathfrak{h}_x$, Δ_x, Δ_x^+ become, respectively, the set of roots of $\mathfrak{c}$ in $\mathfrak{g}$ and in $\mathfrak{n}_x$.

Now for each λ in $\mathfrak{h}^*$ denote by $\mathbb{C}_\lambda$ the corresponding one-dimensional $U(\mathfrak{h})$-module. Let ρ be one-half of the sum of the positive roots. Set

$$\mathscr{D}_\lambda = \mathscr{D}_\mathfrak{h} \otimes_{U(\mathfrak{h})} \mathbb{C}_{\lambda+\rho}.$$

Let ϕ_λ denote the character of $Z(\mathfrak{g})$ obtained by first applying the Harish-Chandra homomorphism γ: $Z(\mathfrak{g}) \to U(\mathfrak{h})$ and then evaluation at $\lambda+\rho$. As is well known, $\phi_\lambda = \phi_{w\lambda}$ for w in the Weyl group, and each character of $Z(\mathfrak{g})$ is obtained in this way [16, Part III; 13, Chap. 7]. This symmetry explains the, unnatural at first, shift by ρ appearing in the above. On the other hand, there are situations where this shift is notationally bothersome. For this reason we will have occasion to use the notation

$$\mathscr{D}_{[\mu]} = \mathscr{D}_\mathfrak{h} \otimes_{U(\mathfrak{h})} \mathbb{C}_\mu. \tag{1.2}$$

Let I_λ be the kernel of ϕ_λ. The following proposition can be deduced from Proposition 1.2 (cf. [3, Sect. 3; 26]):

1.4. PROPOSITION. *The sheaf $\mathscr{D}_\lambda$ is acyclic and locally isomorphic to the sheaf of germs of holomorphic differential operators on X. Its algebra of global sections is isomorphic to $U(\mathfrak{g})/I_\lambda U(\mathfrak{g})$.*

The algebras $\Gamma(\mathscr{D}_\mathfrak{h})$ and $\Gamma(\mathscr{D}_\lambda)$ occur so often in what follows that it is worthwhile giving them special names.

1.5. *Notation.* The algebra $\Gamma(\mathscr{D}_\mathfrak{h}) \cong U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{h})$ will be called the extended enveloping algebra for $\mathfrak{g}$ and denoted $U_\mathfrak{h}(\mathfrak{g})$ or simply $U_\mathfrak{h}$. The algebra $\Gamma(\mathscr{D}_\lambda) \cong U_\mathfrak{h} \otimes_{U(\mathfrak{h})} \mathbb{C}_{\lambda+\rho} \cong U(\mathfrak{g})/U(\mathfrak{g}) I_\lambda$ will be denoted $U_\lambda(\mathfrak{g})$ or simply U_λ.

Note that, as algebras and as $U(\mathfrak{g})$-modules, the U_λ's for λ belonging to the same Weyl group orbit are naturally isomorphic and might, therefore, be given the same name, as they are in [1] where the symbol U_θ is used for this purpose; however, the U_λ's are also $U_{\mathfrak{h}}$-modules and, as such, are not isomorphic, due to the fact that the center $U(\mathfrak{h})$ of $U_{\mathfrak{h}}$ acts by $\lambda+\rho$ on U_λ.

For notational simplicity, we will sometimes let U_λ or $\mathscr{D}_\lambda$ stand for $U_{\mathfrak{h}}$ or $\mathscr{D}_{\mathfrak{h}}$; that is, λ may take the value $\mathfrak{h}$ as well as any value in $\mathfrak{h}^*$.

2. Homology and Localization

We shall be using homological algebra in the context of topological modules over a topological algebra. A suitable theory is worked out in [37, 22]. The theory closely follows that for ordinary modules over ordinary algebras, with the modifications that are forced by the inherent topological difficulties. The main differences are: (1) a completed topological tensor product is used in the definitions of algebras, modules, and tensor and torsion products; (2) the fact that short exact sequences of topological vector spaces do not split mandates the use of a form of relative homological algebra. Below we give a brief discussion of these matters. For background on topological vector spaces and topological tensor products we refer the reader to [31].

The completed projective topological tensor product of two topological vector spaces E and F will be denoted $E \hat{\otimes} F$. A *topological algebra* is an algebra B (over $\mathbb{C}$) with a complete locally convex topological vector space structure and an associative bicontinuous multiplication—that is, one that induces a continuous linear map $B \hat{\otimes} B \to B$. Similarly, if B is a topological algebra, then a (left) *topological B-module* is a complete locally convex topological vector space E with an associative bicontinuous action of B—that is, one that induces a continuous linear map $B \hat{\otimes} E \to E$. If E is a right B-module and F is a left B-module, then the tensor product relative to B, $E \hat{\otimes}_B F$, is the cokernel of the map

$$e \otimes b \otimes f \to eb \otimes f - e \otimes bf\colon E \hat{\otimes} B \hat{\otimes} F \to E \hat{\otimes} F.$$

The resulting space is a topological vector space under the quotient topology, but it need not be Hausdorff, since the above map need not have closed range.

A *free* (*left*) *topological B-module* is, by definition, one of the form $B \hat{\otimes} F$, where B acts through its action on the left factor in the obvious way. A *projective topological B-module* is a topological B-module which is projective relative to the class of surjections which split as morphisms of topological vector spaces. A topological B-module is projective in this sense if and only if it is a topological direct summand of a free topological B-module. Each topological B-module F has a resolution

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to F \to 0$$

by projective topological modules which is $\mathbb{C}$-*split*— that is, split as a sequence of topological vector spaces and continuous maps. If E is a right B-module and we apply the functor $E \hat{\otimes}_B (\ )$ to the complex $\{P_*\}$, we obtain a complex $\{E \hat{\otimes}_B P_*\}$ with $E \hat{\otimes}_B F$ as its zero degree homology; we define $\mathrm{Tor}_p^B(E, F)$ to be its homology in degree p. Any two resolutions as above are homotopy equivalent and so the complex $\{E \hat{\otimes}_B P_*\}$ yields a well-defined element of the homotopy category of complexes of topological vector spaces; thus, $\mathrm{Tor}_p^B(E, F)$ is well defined. Of course, as with ordinary Tor, we get the same functor if we use a projective resolution of the right module E and then tensor with F.

Topological tensor product does not behave well in all circumstances. We want to restrict to algebras and modules which, as topological vector spaces, belong to a category which includes the spaces we wish to study, behaves well under topological tensor product, and has good hereditary properties. A suitable choice is the category of duals of nuclear Fréchet spaces (*DNF spaces*). This category includes all the examples we shall study and is closed under passing to closed subspaces and separated quotients as well as taking projective tensor products and countable inductive limits (provided the limit is Hausdorff). Within this category, the completed projective tensor product is an exact functor in each argument. Furthermore, if E is the inductive limit of a sequence of DNF subspaces $\{E_i\}$ and F is a DNF space, then $E \hat{\otimes} F$ is the inductive limit of the sequence of space $E_i \hat{\otimes} F$. For a more complete discussion of this category see Appendix A. A topological B-module which is DNF as a topological vector space will be called a *DNF B-module*.

Here, we shall be interested in several different algebras B, each of which is of countable dimension. These are not ordinarily thought of as topological algebras—for us, the interesting topology is in the modules, not the algebras. However, the foregoing theory fits this situation if we give each such algebra B the strongest locally convex topology—that is, the inductive limit topology obtained by regarding B as the inductive limit of its finite dimensional subspaces. With this topology, B is DNF and the projective tensor product $B \hat{\otimes} E$ is the inductive limit of spaces of the form $F \hat{\otimes} E = F \otimes E$ for F a finite dimensional subspace of B; i.e., in this case, the algebraic tensor product $B \otimes E$ is already complete in the projective topology and thus coincides with $B \hat{\otimes} E$. We shall be interested specifically in cases where B is one of the algebras $U(\mathfrak{g})$, $U_{\mathfrak{h}}$, or U_λ discussed in Section 1.

Let E and F be DNF $U_{\mathfrak{h}}$-modules; then they are also DNF $U(\mathfrak{g})$-modules and we may consider $\mathrm{Tor}_p^{U(\mathfrak{g})}(E, F)$. As the center of $U_{\mathfrak{h}}$, $U(\mathfrak{h})$ acts as an algebra of endomorphisms on both E and F. As a result, there are two natural $U(\mathfrak{h})$-module structures on $\mathrm{Tor}_p^{U(\mathfrak{g})}(E, F)$—one induced from E and one from F.

We say that a $U_{\mathfrak{h}}$-module has infinitesimal character $\mu \in \mathfrak{h}^*$ if the center $U(\mathfrak{h})$ of $U_{\mathfrak{h}}$ acts on it according to the character $\mu + \rho$.

2.1. PROPOSITION. *Let E be a right DNF $U_{\mathfrak{h}}$-module and let F be a left DNF $U_{\mathfrak{h}}$-module and suppose one of them has regular infinitesimal character. Then, $\mathrm{Tor}_*^{U_{\mathfrak{h}}}(E, F)$ is naturally isomorphic to the direct summand of $\mathrm{Tor}_*^{U(\mathfrak{g})}(E, F)$ consisting of the space on which the natural $U(\mathfrak{h})$-module actions determined by E and F agree.*

Proof. Let us assume for example that F has infinitesimal character λ. In particular the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts on F according to the character ϕ_λ. Consequently, for each $U(\mathfrak{g})$-module E we have $E \hat{\otimes}_{U(\mathfrak{g})} F \cong E' \hat{\otimes}_{U(\mathfrak{g})} F$, where E' is the cokernel of the map $E \oplus Z(\mathfrak{g}) \to E$ defined by $(v, z) \to (z - \phi_\lambda(z))v$. Note that E' has infinitesimal character ϕ_λ when regarded as a right $U(\mathfrak{g})$-module. Suppose now that E is a $U_{\mathfrak{h}}$-module. Then the action of $U(\mathfrak{h})$ on E' factors through $U(\mathfrak{h})/\gamma(\mathrm{Ker}\, \phi_\lambda)$. We recall that $U(\mathfrak{h})/\gamma(\mathrm{Ker}\, \phi_\lambda)$ is a finite dimensional $U(\mathfrak{h})$-module, which, in the case of regular λ, is semisimple, with weights $w\lambda + \rho$, for w in the Weyl group. It follows that E', regarded as a right $U(\mathfrak{h})$-module, splits as a direct sum of modules $E'_{w\lambda}$ with infinitesimal character $w\lambda$. Therefore $\mathrm{Tor}_0^{U_{\mathfrak{h}}}(E, F) \cong E'_\lambda \hat{\otimes}_{U(\mathfrak{h})} F$, which is the subspace of $E \hat{\otimes}_{U(\mathfrak{g})} F \cong \mathrm{Tor}_0^{U_{\mathfrak{g}}}(E, F)$ on which the two actions of $U(\mathfrak{h})$ agree. To prove the general statement, we use the same argument but with E replaced by a $U_{\mathfrak{h}}$-free $\mathbb{C}$-split resolution, which, by the Poincaré–Birkhoff–Witt Theorem is also a $U(\mathfrak{g})$-free $\mathbb{C}$-split resolution.

We shall be interested in these matters primarily in the context of localization. Our notion of localization differs from that of [1] in that we deal with topological modules and use the projective tensor product. The localization functor will be the functor Δ from DNF U_λ-modules to sheaves defined by $\Delta(M) = \mathscr{D}_\lambda \hat{\otimes}_{U_\lambda} M$ (recall that λ can have the value $\mathfrak{h}$ as well as any value in $\mathfrak{h}^*$). To make sense of this, we need to topologize $\mathscr{D}_\lambda$ and define the target category of sheaves.

The topology on $\Gamma(U, \mathscr{D}_\lambda)$ is defined as follows: $\Gamma(U, \mathscr{D}_\lambda)$ is filtered by differential operator degree; for U an affine open set, the module of operators of a given degree is a finite dimensional free $\Gamma(U, \mathcal{O})$-module and, thus, inherits a natural topology from $\Gamma(U, \mathcal{O})$. The topology on $\Gamma(U, \mathscr{D}_\lambda)$ is the inductive limit topology determined by this system of subspaces. If U is not affine, the topology is defined to be the weakest topology for which all restriction maps $\Gamma(U, \mathscr{D}_\lambda) \to \Gamma(V, \mathscr{D}_\lambda)$, for V an affine open subset of U, are continuous.

The standard topology on $\Gamma(U, \mathcal{O})$ is nuclear and Fréchet, but is certainly not DNF. However, let K be a compact set in X and topologize $\Gamma(K, \mathcal{O})$ by representing it as the inductive limit of the system $\{\Gamma(U, \mathcal{O}): U \supset K\}$. Then $\Gamma(K, \mathcal{O})$ is DNF (cf. [14]). In particular, the stalks of $\mathcal{O}$ are DNF spaces with this topology. From the hereditary properties of DNF spaces it follows that $\Gamma(K, \mathcal{D}_\lambda)$ is also a DNF space for each compact set K.

We define $\mathcal{D}_\lambda \hat{\otimes}_{U_\lambda} M$ to be the sheaf determined by the presheaf $V \to \Gamma(V, \mathcal{D}_\lambda) \hat{\otimes}_{U_\lambda} M$, where V ranges over the *compact* subsets of X (see the discussion preceding Proposition 3.3).

2.2. PROPOSITION. *For each x, the stalk $(\mathcal{D}_\lambda \hat{\otimes}_{U_\lambda} M)_x$ is naturally isomorphic to $\mathcal{D}_{\lambda,x} \hat{\otimes}_{U_\lambda} M$, where $\mathcal{D}_{\lambda,x}$ is the stalk of $\mathcal{D}_\lambda$ at x.*

Proof. For each compact V containing x in its interior, we have the exact sequence

$$\Gamma(V, \mathcal{D}_\lambda) \hat{\otimes} M \hat{\otimes} U_\lambda \to \Gamma(V, \mathcal{D}_\lambda) \hat{\otimes} M \to \Gamma(V, \mathcal{D}_\lambda) \hat{\otimes}_{U_\lambda} M \to 0.$$

Since M and U_λ are DNF, $(\cdot) \hat{\otimes} M$ and $(\cdot) \hat{\otimes} M \hat{\otimes} U_\lambda$ as functors on DNF spaces commute with countable inductive limits. Furthermore, the inductive limit functor is right exact. Thus, the proposition follows on passing to the inductive limit over compact sets V with interiors containing x in the above sequence.

The stalk at x of the sheaf $\mathcal{D}_\lambda \hat{\otimes}_{U_\lambda} M$ inherits a $\mathcal{D}_{\lambda,x}$-module structure and a locally convex vector space topology, by virtue of its description as $\mathcal{D}_{\lambda,x} \hat{\otimes}_{U_\lambda} M$. However, in general this topology is not Hausdorff. Thus, $(\mathcal{D}_\lambda \hat{\otimes}_{U_\lambda} M)_x$ may not qualify as a topological $\mathcal{D}_{\lambda,x}$-module. The situation is much better when M is projective, as we shall see in Section 5.

2.3. DEFINITION. We will denote the category of DNF U_λ-modules and continuous homomorphisms by $\mathfrak{M}(U_\lambda)$.

We denote by $\mathfrak{M}^0(\mathcal{D}_\lambda)$ the category whose objects are sheaves of $\mathcal{D}_\lambda$-modules with locally convex topologies on each stalk, and whose morphisms are morphisms of sheaves of modules which are continuous in these locally convex topologies.

We define the localization functor Δ: $\mathfrak{M}(U_\lambda) \to \mathfrak{M}^0(\mathcal{D}_\lambda)$ by $\Delta(M) = \mathcal{D}_\lambda \hat{\otimes}_{U_\lambda} M$.

Proposition 2.1 allows us to compute Δ and its higher derived functors by using the Koszul resolution $K(\mathcal{N}, U(\mathfrak{g}))$ for $\mathcal{D}_\mathfrak{h}$, i.e., the complex (1.1). Recall that this gives a free right $U(\mathfrak{g})$-module resolution of $\mathcal{D}_\mathfrak{h}$, which is locally $\mathbb{C}$-split, by Proposition 1.3. Thus, $\operatorname{Tor}_p^{U(\mathfrak{g})}(\mathcal{D}_\mathfrak{h}, M)$ is the pth homology of the complex $K(\mathcal{N}, U(\mathfrak{g})) \hat{\otimes}_{U(\mathfrak{g})} M$.

As a right $U(\mathfrak{g})$-module, $K_p(\mathcal{N}, U(\mathfrak{g})) \cong (\bigwedge^p \mathcal{N}) \hat{\otimes}_{\mathcal{O}} \mathcal{U} \cong (\bigwedge^p \mathcal{N}) \hat{\otimes} U(\mathfrak{g})$. On applying $(\cdot) \hat{\otimes}_{U(\mathfrak{g})} M$ for a topological $U_{\mathfrak{h}}$-module M, we obtain the sheaf $(\bigwedge^p \mathcal{N}) \hat{\otimes} M$. Thus, the same functor applied to the complex $K(\mathcal{N}, U(\mathfrak{g}))$ yields the Koszul complex $K(\mathcal{N}, M)$. In view of Proposition 2.1, we have established:

2.4. PROPOSITION. *If Δ_p denotes the pth derived functor of Δ and if M is a DNF $U_{\mathfrak{h}}$-module with regular infinitesimal character λ, then there is a natural isomorphism between $\Delta_p(M) = \operatorname{Tor}_p^{U_{\mathfrak{h}}}(\mathcal{D}_{\mathfrak{h}}, M)$ and the subspace of $H_p(K(\mathcal{N}, M))$ on which the $U(\mathfrak{h})$-action induced from $\mathcal{D}_{\mathfrak{h}}$ is that given by the infinitesimal character λ.*

If a $U_{\mathfrak{h}}$-module M has infinitesimal character λ, then it may also be regarded as a U_λ-module. Thus, it appears that we may have two different definitions of $\Delta(M)$ in this case: $\mathcal{D}_\lambda \hat{\otimes}_{U_\lambda} M$ and $\mathcal{D} \hat{\otimes}_{U_{\mathfrak{h}}} M$. Also it appears that $\Delta_p(M)$ could be $\operatorname{Tor}_p^{U_\lambda}(\mathcal{D}_\lambda, M)$ or $\operatorname{Tor}_p^{U_{\mathfrak{h}}}(\mathcal{D}_{\mathfrak{h}}, M)$. However, as we see in the next proposition, there is no ambiguity.

2.5. PROPOSITION. *If M has regular infinitesimal character λ, then there is a natural isomorphism $\operatorname{Tor}_p^{U_\lambda}(\mathcal{D}_\lambda, M) \cong \operatorname{Tor}_p^{U_{\mathfrak{h}}}(\mathcal{D}_{\mathfrak{h}}, M)$ for all p.*

Proof. We have $\mathcal{D}_{\mathfrak{h}} \hat{\otimes}_{U_{\mathfrak{h}}} M \cong \mathbb{C}_\lambda \hat{\otimes}_{U(\mathfrak{h})} \mathcal{D}_{\mathfrak{h}} \hat{\otimes}_{U(\mathfrak{g})} M \cong \mathcal{D}_\lambda \hat{\otimes}_{U(\mathfrak{g})} M \cong \mathcal{D}_\lambda \hat{\otimes}_{U_\lambda} M$ and so $\operatorname{Tor}_p^{U_\lambda}(\mathcal{D}_\lambda, \cdot)$ is naturally isomorphic to $\operatorname{Tor}_p^{U_{\mathfrak{h}}}(\mathcal{D}_{\mathfrak{h}}, \cdot)$ for $p = 0$ as functors on the category of topological $U_{\mathfrak{h}}$-modules with infinitesimal character λ. Note that $\operatorname{Tor}_p^{U_\lambda}(\mathcal{D}_\lambda, \cdot)$ vanishes on free U_λ-modules for $p > 0$. We shall show that $\operatorname{Tor}_p^{U_{\mathfrak{h}}}(\mathcal{D}_{\mathfrak{h}}, \cdot)$ also vanishes on free U_λ-modules for $p > 0$. The proposition then follows from a standard induction argument using the long exact sequences for $\operatorname{Tor}_p^{U_\lambda}(\mathcal{D}_\lambda, \cdot)$ and $\operatorname{Tor}_p^{U_{\mathfrak{h}}}(\mathcal{D}_{\mathfrak{h}}, \cdot)$ applied to the $\mathbb{C}$-split short exact sequence

$$0 \to K \to U_\lambda \hat{\otimes} M \to M \to 0,$$

where $U_\lambda \hat{\otimes} M \to M$ is the multiplication map and K is its kernel.

It remains, then, to show that $\operatorname{Tor}_p^{U_{\mathfrak{h}}}(\mathcal{D}_{\mathfrak{h}}, \cdot)$ kills free U_λ-modules. If $U_\lambda \hat{\otimes} E$ is such a module, then Proposition 2.4 tells us that $\operatorname{Tor}_p^{U_{\mathfrak{h}}}(\mathcal{D}_{\mathfrak{h}}, U_\lambda \hat{\otimes} E)$ is a subspace of the homology module $H_p(K(\mathcal{N}, U_\lambda \hat{\otimes} E)) \cong H_p(K(\mathcal{N}, U_\lambda) \hat{\otimes} E)$. Since $(\cdot) \hat{\otimes} E$ is an exact functor on DNF spaces, this last module is isomorphic to $H_p(K(\mathcal{N}, U_\lambda)) \hat{\otimes} E$. Thus, it suffices to show that $H_p(K(\mathcal{N}, U_\lambda))$ vanishes for $p > 0$. However, this follows from the (well-known) fact that U_λ is free as a $U(\mathfrak{n}_x)$-module for each x (cf. [25]) and the fact that, as in the proof of Proposition 1.3, the complex of bundles $K(\mathfrak{N}, U_\lambda)$ is locally trivial.

3. Sheaves of Modules

The result we are aiming for involves an equivalence between a derived category for the category $\mathfrak{M}(U_\lambda)$ of DNF U_λ-modules and a derived category for a certain category $\mathfrak{M}(\mathscr{D}_\lambda)$ of sheaves of $\mathscr{D}_\lambda$-modules. In this section we describe this category of sheaves and some of its properties. Although most of the results hold in a more general context, we choose to present them in this form to avoid unnecessary notational complications. The generalizations are straightforward.

3.1. Definition. A DNF sheaf of $\mathscr{D}_\lambda$-modules will be a sheaf $\mathscr{M}$ of $\mathscr{D}_\lambda$-modules with the following additional structure: for each compact set K, $\Gamma(K, \mathscr{M})$ has the structure of a DNF topological $\Gamma(K, \mathscr{D}_\lambda)$-module in such a way that, for each nested pair of compact sets $K \subset L$, the restriction map $\Gamma(L, \mathscr{M}) \to \Gamma(K, \mathscr{M})$ is continuous. We shall denote by $\mathfrak{M}(\mathscr{D}_\lambda)$ the category whose objects are DNF sheaves of $\mathscr{D}_\lambda$-modules and whose morphisms are those homomorphisms of $\mathscr{D}_\lambda$-modules which induce continuous homomorphisms between the modules of sections over compact sets.

Points are compact sets; so if $\mathscr{M}$ is a DNF sheaf of $\mathscr{D}_\lambda$-modules, then $\mathscr{M}_x$ is a DNF $\mathscr{D}_{\lambda,x}$-module for each $x \in X$. It turns out that, for DNF sheaves of modules, the topological vector space structure on the stalks determines that on the sections over any compact set.

3.2. Proposition. *If $\mathscr{M}$ and $\mathscr{N}$ are DNF sheaves of $\mathscr{D}_\lambda$-modules and $\phi\colon \mathscr{M} \to \mathscr{N}$ is a sheaf homomorphism, then ϕ is a morphism in the category $\mathfrak{M}(\mathscr{D}_\lambda)$, provided the map $\phi_x\colon \mathscr{M}_x \to \mathscr{N}_x$ is a continuous $\mathscr{D}_{\lambda,x}$-module homomorphism for each $x \in X$. If ϕ_x is a continuous isomorphism for each x, then ϕ is an isomorphism in the category $\mathfrak{M}(\mathscr{D}_\lambda)$.*

Proof. Let K be a compact subset of X. Since both $\Gamma(K, \mathscr{M})$ and $\Gamma(K, \mathscr{N})$ are DNF spaces, the closed graph theorem applies (cf. Appendix A, A.7). If $(f, g) \in \Gamma(K, \mathscr{M}) \times \Gamma(K, \mathscr{N})$ is a point in the closure of the graph of ϕ: $\Gamma(K, \mathscr{M}) \to \Gamma(K, \mathscr{N})$, then $(f(x), g(x))$ belongs to the graph of $\phi_x\colon \mathscr{M}_x \to \mathscr{N}_x$, since the restriction maps and ϕ_x are continuous by hypothesis. That is, $f(x) = \phi_x g(x)$ for each x, from which we conclude that $f = \phi g$ or (f, g) belongs to the graph of ϕ. By the closed graph theorem, ϕ: $\Gamma(K, \mathscr{M}) \to \Gamma(K, \mathscr{N})$ is continuous. Thus, ϕ is a morphism in $\mathfrak{M}(\mathscr{D}_\lambda)$.

If each ϕ_x is a continuous isomorphism, then it is also a topological isomorphism by the open mapping theorem (cf. Appendix A, A.6) and so the result of the first paragraph applies to the inverse of ϕ as well.

In constructing a sheaf from a presheaf we need not use all open sets.

Actually, we do not even need to use open sets. For example, let $\mathscr{U}$ be a collection of closures of a certain fixed basis for the topology on X. By a *presheaf of topological vector spaces based on* $\mathscr{U}$ we shall mean a correspondence S which assigns to each $U \in \mathscr{U}$ a topological vector space $S(U)$ and to each nested pair $U \subset V$ in $\mathscr{U}$ a continuous homomorphism $r_{V,U}: S(U) \to S(V)$. The sheaf $\mathscr{S}$ generated by a presheaf in this sense is defined in the usual way. Each stalk $\mathscr{S}_x = \lim\{S(U): U \in \mathscr{U}$ and $x \in$ interior of $U\}$ is a topological vector space under the inductive limit topology. This topology may fail to be Hausdorff.

3.3. PROPOSITION. *Let S be a presheaf of topological vector spaces based on $\mathscr{U}$ as above. If $S(U)$ is a (not necessarily separated) quotient of a DNF space for each $U \in \mathscr{U}$ and if each stalk of $\mathscr{S}$ is Hausdorff in the inductive limit topology, then the sheaf $\mathscr{S}$ generated by S has a unique structure of a DNF sheaf, compatible with the topological structure on the stalks.*

Proof. Assume first that $S(U)$ is DNF for each $U \in \mathscr{U}$. We must describe a DNF structure on $\Gamma(K, \mathscr{S})$ for each compact subset K of X. By a *cover* of K we mean a collection of sets in $\mathscr{U}$ such that K is contained in the union of their interiors. Clearly every cover of K has a finite subcover. For a finite cover $\mathscr{F} = \{U_1, ..., U_n\}$ of K we define a DNF space

$$S_{\mathscr{F}} = \left\{ \{s_i\} \in \bigoplus_i S(U_i): r_{U_j \cap U_k, U_j}(s_j) \right.$$
$$\left. = r_{U_j \cap U_k, U_k}(s_k) \text{ for } j, k = 1, ..., n \right\}.$$

Given two finite subsets $\mathscr{F}$ and $\mathscr{G}$ of K, with $\mathscr{G}$ a refinement of $\mathscr{F}$, the restriction maps clearly define a continuous homomorphism from $S_{\mathscr{F}}$ to $S_{\mathscr{G}}$. The finite subsets of $\mathscr{U}$ which cover K form a directed set and the collection $\{S_{\mathscr{F}}\}$ is a directed system indexed over this set. It is evident that $\Gamma(K, \mathscr{S})$ is the limit of this system. We use this description to define a topology on $\Gamma(K, \mathscr{S})$. The direct limit topology on $\Gamma(K, \mathscr{S})$ is Hausdorff—this follows from the conditions on stalks—and hence DNF since the system $\{S_{\mathscr{F}}\}$ has a countable cofinal subsystem. Given a nested pair of compact sets $K \supset L$, the restriction map $\Gamma(K, \mathscr{S}) \to \Gamma(L, \mathscr{S})$ is clearly continuous under the resulting topologies on $\Gamma(K, \mathscr{S})$ and $\Gamma(L, \mathscr{S})$. Thus, $\mathscr{S}$ is a DNF sheaf. The uniqueness of the DNF structure follows immediately from Proposition 3.2.

Now, let us drop the assumption that $S(U)$ is DNF. Let $S'(U)$ be the quotient of $S(U)$ modulo the closure of $\{0\}$. Then S' forms a presheaf of DNF spaces. Since, by assumption, the stalks of $\mathscr{S}$ are Hausdorff, the homomorphism $S(U) \to \mathscr{S}_x$ factors through $S(U) \to S'(U)$ for each $x \in X$. It follows that the natural homomorphism of presheaves $S \to S'$ induces an isomorphism on the associated sheaves, and we define the topology on $\mathscr{S}$ using S', in the manner described above.

3.4. COROLLARY. *If M is a DNF U_λ-module, then $\Delta_p(M)$ is a DNF sheaf of $\mathscr{D}_\lambda$-modules whenever it has Hausdorff stalks.*

Proof. By definition, $\Delta_p(M)$ is the pth homology sheaf of the complex $\mathscr{P}$ of free $\mathscr{D}_\lambda$-modules, obtained by taking a complete tensor product of $\mathscr{D}_\lambda$ with a free resolution of M. Let $\mathscr{U}$ be as above. Then $\Delta_p(M)$ is the sheaf associated to the presheaf S of quotients of DNF spaces, which assigns to each $U \in \mathscr{U}$ the pth homology group of the complex $\Gamma(U, \mathscr{P})$. The corollary now follows from Proposition 3.3.

It is also evident that the topological structure of $\Delta_p(M)$ is independent of the free resolution of M. We get the same structure of a DNF sheaf of $\mathscr{D}_\lambda$-modules on $\Delta_p(M)$, if we compute it from the Koszul resolution, as described in Propositions 2.4 and 2.5.

In Section 5, when we establish our result on equivalence of derived categories, we shall need to know that sheaves in $\mathfrak{M}(\mathscr{D}_\lambda)$ have resolutions by sheaves which are acyclic in a very strong sense, and still belong to $\mathfrak{M}(\mathscr{D}_\lambda)$. The following construction, which is just a disguised version of Čech theory, gives us the resolutions we need.

Let π: $Y \to X$ be a surjection from a compact, totally disconnected, metric space Y onto X. If $\mathscr{M}$ is a sheaf on X, then its pullback to Y, $\pi^{-1}\mathscr{M}$, is a sheaf of modules on Y with the property that its stalk at $y \in Y$ is canonically isomorphic to the stalk of $\mathscr{M}$ at $\pi(y)$. If $\mathscr{M}$ is a DNF sheaf of $\mathscr{D}_\lambda$-modules, we topologize $\Gamma(K, \pi^{-1}\mathscr{M})$ for K compact in Y as follows: For a given finite cover $C = \{K_i\}$ of K by compact subsets, let $\Gamma_C(K, \pi^{-1}\mathscr{M})$ be the space of sections f with the property that, for each i, $f|_{K_i}$ has the form $g_i \circ \pi$ for some section $g_i \in \Gamma(\pi(K_i), \mathscr{M})$. Then $\Gamma_C(K, \pi^{-1}\mathscr{M})$ is naturally a closed submodule of $\bigoplus_i \Gamma(\pi(K_i), \mathscr{M})$ and, hence, a DNF module. It is straightforward to show that $\Gamma(K, \pi^{-1}\mathscr{M})$ is the direct limit of a countable family of such submodules. The resulting direct limit DNF topology on $\Gamma(K, \pi^{-1}\mathscr{M})$ is the unique one for which each restriction map $\Gamma(K, \pi^{-1}\mathscr{M}) \to (\pi^{-1}\mathscr{M})_y = \mathscr{M}_{\pi(y)}$ is continuous for each y.

Because Y is compact and totally disconnected, each open cover of Y has a refinement which is a finite decomposition into disjoint compact–open sets. Given such a decomposition $\{U_i\}$, the corresponding set of restriction maps $\{r_i\}$ provides a partition of unity for $\pi^{-1}\mathscr{M}$ consisting of continuous $\pi^{-1}\mathscr{D}_\lambda$-module homomorphisms. Evidently, the direct image $\pi_* \pi^{-1}\mathscr{M}$, which we will denote from now on by $\mathscr{C}^0(\mathscr{M})$ is again DNF, and is fine.

Actually, it is also $\mathscr{D}_\lambda$-*fine*; i.e., subordinate to each open cover of X there is a partition of unity consisting of morphisms in $\mathfrak{M}(\mathscr{D}_\lambda)$.

If Y^{p+1} denotes the fiber product over X of Y with itself $p+1$ times, then we may repeat the above construction for the surjection $\pi_p\colon Y^{p+1} \to X$. The direct image of the pullback of $\mathscr{M}$ relative to this map we denote by $\mathscr{C}^p(\mathscr{M})$. As before, this is a $\mathscr{D}_\lambda$-fine sheaf of DNF modules if $\mathscr{M}$ is a DNF sheaf of $\mathscr{D}_\lambda$-modules. We set

$$\pi_{p,j}(y_0, ..., y_{p+1}) = (y_0, ..., y_{j-1}, y_{j+1}, ..., y_{p+1})$$

to obtain $p+1$ projections of Y^{p+1} onto Y^p. Each of these projections induces a natural transformation of functors $\pi_{p*}\pi_p^{-1} \to \pi_{p*}(\pi_{p,j})_*(\pi_{p,j}^{-1})\pi_p^{-1} \cong (\pi_{p+1})_*(\pi_{p+1})^{-1}$ and, therefore, also a map $\mathscr{C}^{p-1}(\mathscr{M}) \to \mathscr{C}^p(\mathscr{M})$. The alternating sum of these defines a differential $d^{p-1}\colon \mathscr{C}^{p-1}(\mathscr{M}) \to \mathscr{C}^p(\mathscr{M})$ for each p. This, together with the natural embedding $d^{-1}\colon \mathscr{M} \to \mathscr{C}^0(\mathscr{M})$, yields a complex

$$0 \to \mathscr{M} \to \mathscr{C}^0(\mathscr{M}) \to \mathscr{C}^1(\mathscr{M}) \to \cdots. \tag{3.1}$$

We shall call the complex $\{\mathscr{C}^*(\mathscr{M})\}$ the *Čech complex* of $\mathscr{M}$ and denote it by $\mathscr{C}(\mathscr{M})$. We have:

3.5. PROPOSITION. *For each DNF sheaf of $\mathscr{D}_\lambda$-modules $\mathscr{M}$, the Čech complex provides a resolution of $\mathscr{M}$ by $\mathscr{D}_\lambda$-fine sheaves of DNF $\mathscr{D}_\lambda$-modules.*

Proof. All that remains is to show that (3.1) is exact. We do this by showing that, for a given x in X, the complex of stalks at x of (3.1) has a contracting homotopy γ. If we regard elements $f \in \mathscr{C}^p(\mathscr{M})_x$ as locally constant $\mathscr{M}_x$-valued functions on $\pi_p^{-1}(x) = \pi^{-1}(x) \times \cdots \times \pi^{-1}(x)$ $(p+1$ times)—we may do it, since π_p is a proper map—then the proof of this is quite standard: We fix $y' \in \pi^{-1}(x)$ and, for $f \in \mathscr{C}^p(\mathscr{M})_x$, set

$$\gamma_p f(y_0, ..., y_{p-1}) = f(y', y_0, ..., y_{p-1})$$

if $p > 0$. Then $\gamma_p f \in \mathscr{C}^{p-1}(\mathscr{M})_x$. We also set $\gamma_0 f = f(y') \in \mathscr{M}_x$. Then

$$\begin{aligned}
&(\gamma_{p+1} d^p + d^{p-1}\gamma_p) f(y_0, ..., y_p) \\
&\quad = f(y_0, ..., y_p) - \sum_{j=0}^{p} (-1)^j f(y', y_0, ..., y_{j-1}, y_{j+1}, ..., y_p) \\
&\qquad + \sum_{j=0}^{p} (-1)^j f(y', y_0, ..., y_{j-1}, y_{j+1}, ..., y_p) \\
&\quad = f(y_0, ..., y_p).
\end{aligned}$$

Thus, γ is a contracting homotopy for (3.1) over the point x and, since we can do this at each x, (3.1) is an exact sequence of sheaves.

An important operation on the category of DNF sheaves is that of tensoring with a DNF space. Thus, if $\mathcal{S}$ is DNF we define $\mathcal{S} \hat{\otimes} E$ to be the sheaf associated to the presheaf $\Gamma(U, \mathcal{S}) \hat{\otimes} E$ based on the family $\mathcal{U}$ of compact neighborhoods in X. The exactness properties of $\hat{\otimes}$ imply the following:

3.6. LEMMA (Compare [30, Proposition 5(ii)]). *Let $\mathcal{S}$ be a DNF sheaf and E a DNF space. Then, for each compact set K in X, $\Gamma(K, \mathcal{S} \hat{\otimes} E) \cong \Gamma(K, \mathcal{S}) \hat{\otimes} E$.*

4. A VANISHING THEOREM

In the next section we shall prove our general equivalence of categories result. The key step in that proof is the following vanishing theorem for the localization functor:

4.1. THEOREM. *For $x \in X$ and λ a regular element of $\mathfrak{h}^*$, let M be a DNF module over the stalk $\mathcal{D}_{\lambda,x}$ of $\mathcal{D}_\lambda$ at x. Then $\Delta_p(M) = 0$ if $p > 0$ and $\Delta(M)$ is the sheaf supported at x with stalk M.*

The proof involves a series of lemmas.

4.2. LEMMA. *If Theorem 4.1 is true for the case $M = \mathcal{D}_{\lambda,x}$, then it is true in general.*

Proof. The theorem is true for all free modules if it is true when $M = \mathcal{D}_{\lambda,x}$, since a free DNF $\mathcal{D}_{\lambda,x}$-module has the form $\mathcal{D}_{\lambda,x} \hat{\otimes} E$ for a DNF space E, and $(\cdot) \hat{\otimes} E$ is an exact functor on DNF spaces.

Let i^x: $\{x\} \to X$ be the inclusion and regard $\mathcal{D}_{\lambda,x}$-modules as sheaves over the singleton point $\{x\}$. Then $i^x_* M$ is a DNF $\mathcal{D}_\lambda$-module supported on $\{x\}$ with stalk M at x. Let Δ^x be the restriction of Δ to the category of $\mathcal{D}_{\lambda,x}$-modules. The composition

$$\Delta^x(M) = \mathcal{D}_\lambda \hat{\otimes}_{U_\lambda} M \to \mathcal{D}_\lambda \hat{\otimes}_{U_\lambda} i^x_* M \to \mathcal{D}_\lambda \hat{\otimes}_{\mathcal{D}_\lambda} i^x_* M = i^x_* M,$$

defines a natural transformation of functors $\Delta^x \to i^x_*$. The content of the theorem is that this transformation is an isomorphism and, when restricted to the category of $\mathcal{D}_{\lambda,x}$-modules, the higher derived functors of Δ vanish.

We may assume the theorem is true for free $\mathcal{D}_{\lambda,x}$-modules. Let

$$\cdots \to F_1 \to F_0 \to M \to 0$$

be a $\mathbb{C}$-split resolution of M by free $\mathscr{D}_{\lambda,x}$-modules. If we apply the transformation to this sequence we have a commutative diagram

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \Delta^x(F_1) & \longrightarrow & \Delta^x(F_0) & \longrightarrow & \Delta^x(M) & \longrightarrow & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \\ \cdots \longrightarrow & i^x_*(F_1) & \longrightarrow & i^x_*(F_0) & \longrightarrow & i^x_*(M) & \longrightarrow & 0 \end{array} \tag{4.1}$$

and, by hypothesis, all the vertical morphisms are isomorphisms except possibly the last one. However, i^x_* is exact and Δ^x is right exact, from which it follows that the last vertical map is also an isomorphism.

We now have that the functors Δ^x and i^x_* are isomorphic. Since i^x_* is exact, so is Δ^x. Therefore the top horizontal sequence in (4.1) is exact. Since, by assumption, each F_k is Δ-acyclic, we conclude that the functors Δ_p vanish for all $\mathscr{D}_{\lambda,x}$-modules M when $p>0$.

By Propositions 2.4 and 2.5, we compute $\Delta_p(\mathscr{D}_{\lambda,x})$ by passing to the appropriate summand of the pth homology of the Koszul complex of sheaves $K(\mathscr{N}, \mathscr{D}_{\lambda,x})$.

4.3. Lemma. *For each $y \in X$, the stalk $K(\mathscr{N}, \mathscr{D}_{\lambda,x})_y$ of the Koszul complex is isomorphic, as a complex of topological $\mathscr{O}_y$-modules, to $\mathscr{O}_y \mathbin{\hat{\otimes}} K(\mathfrak{n}_y, \mathscr{D}_{\lambda,x})$.*

Proof. We shall use the adjoint of the G-action on X to locally trivialize the Koszul complex. Let $p\colon G \to X$ be the projection $g \to gy$ and, over a neighborhood U of y, let $\gamma\colon U \to G$ be a section of p with $\gamma(y)=e$. Then for $y' \in U$, $\mathrm{Ad}(\gamma(y'))$ maps $\mathfrak{n}_y$ to $\mathfrak{n}_{y'}$. This induces an isomorphism of $\mathscr{O}|_U$-modules $\gamma\colon \mathscr{O}|_U \mathbin{\hat{\otimes}} \mathfrak{n}_y \to \mathscr{N}|_U$. If $f\colon U \to \mathfrak{n}_y$ is an element of $\mathscr{O}|_U \mathbin{\hat{\otimes}} \mathfrak{n}_y$, and we regard $\mathscr{N}$ as the sheaf of sections of the vector bundle $\mathfrak{N}$, then γ is given by the formula

$$\gamma(f)(y') = \mathrm{Ad}(\gamma(y'))\, f(y'). \tag{4.2}$$

However, $\mathrm{Ad}(\gamma(y'))$ does not leave $\mathscr{D}_{\lambda,x}$ invariant, but transforms it to $\mathscr{D}_{\lambda,\gamma(y')x}$. This appears to be a problem, but is not, as we see below.

The point is that $\mathscr{O}_y \mathbin{\hat{\otimes}} \mathscr{D}_{\lambda,x}$ is the stalk at (x, y) of the sheaf $\pi_1^*\mathscr{D}_\lambda$ ($\pi_1, \pi_2\colon X \times X \to X$ denote the projections on the first and second factor, respectively), and $\mathscr{D}_\lambda$ is G-homogeneous. Let us examine this more carefully.

The group G acts on the sheaf of algebras $U(\mathscr{G}) = \mathscr{O} \otimes U(\mathfrak{g})$ by the formula $f \otimes v \to \tau(g) f \otimes \mathrm{Ad}(g)v$ for $g \in G$, where τ is induced by the action of G on X. This also induces an action on $\mathscr{D}_\lambda$, which is a quotient of $U(\mathscr{G})$ (see Sect. 1). Therefore the "shear transformation" $s\colon (x', y') \to (\gamma(y')x', y')$ acts as a transformation S on $\pi_1^*\mathscr{D}_\lambda$, restricted to $\pi_2^{-1}(U)$. It is important to understand the explicit formula for this action. Every element η in $\pi_1^*\mathscr{D}_\lambda$ is represented by $\sum_k f_k \otimes v_k \in \mathcal{O}_{X\times X} \otimes U(\mathfrak{g})$. Set $\delta_{(x',y')} = \sum_k f_k(x', y') \otimes v_k \in U(\mathfrak{g})$. Then $S\eta$ is represented by $\delta' \in \mathcal{O}_{X\times X} \otimes U(\mathfrak{g})$ defined by

$$\delta'_{s(x',y')} = \mathrm{Ad}(\gamma(y'))\,\delta_{(x',y')}. \tag{4.3}$$

Since s has (x, y) as a fixed point, S acts on the stalk $\mathcal{O}_y \hat{\otimes} \mathscr{D}_{\lambda,x}$. The maps γ and S induce an $\mathcal{O}_y$-module isomorphism

$$(\gamma \hat{\otimes} S)\colon \mathcal{O}_y \hat{\otimes} \textstyle\bigwedge^p \mathfrak{n}_y \hat{\otimes} \mathscr{D}_{\lambda,x} \cong (\bigwedge^p \mathfrak{n}_y \hat{\otimes} \mathcal{O}_y) \hat{\otimes}_{\mathcal{O}} (\mathcal{O}_y \hat{\otimes} \mathscr{D}_{\lambda,x})$$

$$\to \textstyle\bigwedge^p \mathscr{N}_y \hat{\otimes} \mathscr{D}_{\lambda,x}.$$

Formulas (4.2) and (4.3) imply that, for $f \in \mathcal{O}_y \hat{\otimes} \mathfrak{n}_y$ and $\eta \in \mathcal{O}_y \hat{\otimes} \mathscr{D}_{\lambda,x}$, $\gamma(f)\,S(\eta) = (\gamma \hat{\otimes} S)(f\eta)$. It follows that $(\gamma \hat{\otimes} S)$ induces an isomorphism between the Koszul complexes $\mathcal{O}_y \hat{\otimes} K(\mathfrak{n}_y, \mathscr{D}_{\lambda,x})$ and $K(\mathscr{N}, \mathscr{D}_{\lambda,x})_y$.

In order to analyze $K(\mathfrak{n}_y, \mathscr{D}_{\lambda,x})$, we represent $\mathscr{D}_{\lambda,x}$ as $U_\lambda \hat{\otimes}_{U_\lambda} \mathscr{D}_{\lambda,x}$, so that $K(\mathfrak{n}_y, \mathscr{D}_{\lambda,x})$ becomes $K(\mathfrak{n}_y, U_\lambda) \hat{\otimes}_{U_\lambda} \mathscr{D}_{\lambda,x}$. As we remarked before, U_λ is free as a $U(\mathfrak{n}_y)$-module and so the Koszul complex $K(\mathfrak{n}_y, U_\lambda)$ is exact except in degree zero and its degree zero homology is the right U_λ-module $V_y = U_\lambda/\mathfrak{n}_y U_\lambda$. Thus, the Koszul complex $K(\mathfrak{n}_y, U_\lambda)$ provides a free right U_λ-module resolution of V_y. Applying $(\cdot) \hat{\otimes}_{U_\lambda} \mathscr{D}_{\lambda,x}$ to such a resolution yields a complex whose homology is $\mathrm{Tor}_*^{U_\lambda}(V_y, \mathscr{D}_{\lambda,x})$. Thus, the homology of $K(\mathfrak{n}_y, \mathscr{D}_{\lambda,x})$ is naturally isomorphic to $\mathrm{Tor}_*^{U_\lambda}(V_y, \mathscr{D}_{\lambda,x})$.

Now, V_y is also a left $U(\mathfrak{h})$-module and, in fact, $U(\mathfrak{h})$ acts semisimply on V_y with weights $\{w\lambda + \rho\colon w \in W\}$, where W is the Weyl group. Let $V_{\lambda,y}$ denote the summand with weight $\lambda + \rho$. Since the left action of $U(\mathfrak{h})$ on V_y commutes with the right action of U_λ, $V_{\lambda,y}$ is a U_λ-module direct summand of V_y. It follows that the action of $U(\mathfrak{h})$ on V_y induces a similar action on $\mathrm{Tor}_*^{U_\lambda}(V_y, \mathscr{D}_{\lambda,x})$ and that the weight space with weight $\lambda+\rho$ is $\mathrm{Tor}_*^{U_\lambda}(V_{\lambda,y}, \mathscr{D}_{\lambda,x})$. Now, we may clearly reverse the roles of right and left modules in our previous discussion of localization. With this done, $V_\lambda \hat{\otimes}_{U_\lambda} \mathscr{D}_\lambda$ is the localization $\Delta(V_{\lambda,y})$ of the right U_λ-module V_λ, while $\mathrm{Tor}_p^{U_\lambda}(V_{\lambda,y}, \mathscr{D}_\lambda)$ is the pth derived functor $\Delta_p(\cdot)$ applied to $V_{\lambda,y}$. Thus, we have proved:

4.4. Lemma. *In the pth homology of $K(\mathfrak{n}_y, \mathscr{D}_{\lambda,x})$, the weight space with weight $\lambda+\rho$ is naturally isomorphic to $\Delta_p(V_{\lambda,y})_x$.*

With this in mind, the proof of Theorem 4.1 will follow easily from the following lemma:

4.5. LEMMA. *The sheaf $\Delta(V_{\lambda,y})$ is the sheaf supported at y with stalk $V_{\lambda,y}$, while $\Delta_p(V_{\lambda,y})=0$ for $p>0$.*

Proof. Note that $V_{\lambda,y}$ has the strongest locally convex topology and so $V_{\lambda,y}\,\hat{\otimes}_{U_\lambda}\mathscr{D}_\lambda$ is, in this case, the algebraic tensor product $V_{\lambda,y}\otimes_{U_\lambda}\mathscr{D}_\lambda$ while $\operatorname{Tor}_p^{U_\lambda}(V_{\lambda,y},\mathscr{D}_\lambda)$ is the algebraic torsion product (cf. Sect. 2). Furthermore, $\mathscr{D}_\lambda$ is isomorphic to $(\mathscr{D}_\lambda^{\text{alg}})^{\text{an}}$, where $\mathscr{D}_\lambda^{\text{alg}}$ is the algebraic analogue of $\mathscr{D}_\lambda$ as introduced in [1], and $\mathscr{R}$ is the sheaf of regular functions and $(-)^{\text{an}}$ is the functor defined in [36]. It follows from this and [36, Sect. 6, Corollaire 1] that $\Delta_p(V_{\lambda,y})\cong(\Delta_p^{\text{alg}}(V_{\lambda,y}))^{\text{an}}$, where Δ_p^{alg} is the pth derived functor of the algebraic localization Δ^{alg} (cf. [2, 25]).

Set $\mathscr{F}=\mathscr{D}_\lambda^{\text{alg}}/\mathscr{I}_y\mathscr{D}_\lambda^{\text{alg}}$, where $\mathscr{I}_y$ is the ideal sheaf in $\mathscr{R}$ corresponding to the variety $\{y\}$ and note that $\mathscr{F}$ is the sheaf supported at y with stalk $V_{\lambda,y}$ at y (compare [3, Sect. 3]). As a right $\mathscr{R}$-module $\mathscr{D}_\lambda^{\text{alg}}$ is quasicoherent and it follows that $\mathscr{F}$ is as well. Since it is supported at a point, $\mathscr{F}$ is an acyclic sheaf. At this point we apply the derived functor version of the equivalence of categories result of Beilinson and Bernstein: Assume that λ is regular. If $R\Gamma$ denotes the right derived functor of the global section functor Γ, then $R\Gamma$ is an equivalence of categories from the derived category of the category of quasicoherent right $\mathscr{D}_\lambda^{\text{alg}}$-modules to the derived category of the category of right $\Gamma(\mathscr{D}_\lambda^{\text{alg}})$-modules; the inverse functor is the left derived functor of Δ^{alg} (cf. 2, Sect. 13; 25]). Since $\mathscr{F}$ has vanishing higher cohomology, $R\Gamma(\mathscr{F})$ is the natural image in the derived category of the right U_λ-module $\Gamma(\mathscr{F})$, which is isomorphic to $V_{\lambda,y}$. It follows that the right derived functor of localization sends $V_{\lambda,y}$ to the natural image in the derived category of the sheaf $\mathscr{F}$. Hence, $\Delta^{\text{alg}}(V_{\lambda,y})$ is the sheaf supported at y with stalk $V_{\lambda,y}$, while $\Delta_p^{\text{alg}}(V_{\lambda,y})=0$ for $p>0$. In view of the comments in the first paragraph, the same things are true of $\Delta(V_{\lambda,y})$ and $\Delta_p(V_{\lambda,y})$.

The proof of Theorem 4.1 is now essentially complete. Lemma 4.2 reduces the problem to the case $M=\mathscr{D}_{\lambda,x}$. That $\Delta(\mathscr{D}_{\lambda,x})$ is supported at x and that $\Delta_p(\mathscr{D}_{\lambda,x})=0$ for $p>0$ both follow directly from Lemmas 4.3, 4.4, and 4.5. To complete the proof, we consider the maps

$$\mathscr{D}_{\lambda,x}\xrightarrow{\alpha}\mathscr{D}_{\lambda,x}\,\hat{\otimes}_{U_\lambda}\mathscr{D}_{\lambda,x}\xrightarrow{\beta}\mathscr{O}_x\,\hat{\otimes}\,V_{\lambda,x},$$

where α is the map $\xi\to 1\,\hat{\otimes}\,\xi$ and β is the isomorphism induced on zero homology by the trivialization of the Koszul complex in Lemma 4.3. By construction, α is a left U_λ-module homomorphism and β is a left $\mathscr{O}_x$-module isomorphism. Recall from Section 1 that $\mathscr{D}_\lambda=\mathbb{C}_{\lambda+\rho}\otimes_{U(\mathfrak{h})}\mathscr{D}_{\mathfrak{h}}$ and that $\mathscr{D}_{\mathfrak{h}}$ may be regarded as the local sections of the holomorphic bundle with fiber $U(\mathfrak{g})/\mathfrak{n}_y U(\mathfrak{g})$ over y. It follows that, as a sheaf of right $U(\mathfrak{g})$-modules, $\mathscr{D}_\lambda$ is the sheaf of local sections of the holomorphic bundle with fiber equal to $V_{\lambda,x}$—the $(\lambda+\rho)$-weightspace of $U_\lambda/\mathfrak{n}_y U_\lambda$ as a right $U(\mathfrak{h})$-module. The composition of α and β is just the local trivialization of this bundle near x given by conjugating by the section γ from Lemma 4.3. Thus, $\beta\circ\alpha$ is a left $\mathcal{O}_x$-module isomorphism. We conclude that α is also a left $\mathcal{O}_x$-module isomorphism and, thus, a left $\mathcal{D}_{\lambda,x}$-module isomorphism since it is also U_λ-invariant, and $\mathcal{O}_x$ and U_λ generate $\mathcal{D}_{\lambda,x}$. Of course, it is obvious from the construction that α is also a right $\mathcal{D}_{\lambda,x}$-module isomorphism. This completes the proof of Theorem 4.1.

4.6. *Remark.* The analogue of Theorem 4.1 does not hold in the algebraic category. The point is that the transformation S of Lemma 4.3 does not act on $\mathcal{R}_y\otimes(\mathcal{D}_\lambda^{\mathrm{alg}})_x$.

5. Equivalence of Categories

In Section 2 we defined the category $\mathfrak{M}(U_\lambda)$ of DNF U_λ-modules and the localization functor Δ: $\mathfrak{M}(U_\lambda)\to\mathfrak{M}^0(\mathcal{D}_\lambda)$, where $\mathfrak{M}^0(\mathcal{D}_\lambda)$ is the category whose objects are sheaves of $\mathcal{D}_\lambda$-modules with (not necessarily Hausdorff) topological vector space structures on stalks and whose morphisms are homomorphisms of $\mathcal{D}_\lambda$-modules which are also continuous on stalks. In Section 3 we defined the category $\mathfrak{M}(\mathcal{D}_\lambda)$ of DNF sheaves of $\mathcal{D}_\lambda$-modules. Note that, by Proposition 3.2, $\mathfrak{M}(\mathcal{D}_\lambda)$ is a full subcategory of $\mathfrak{M}^0(\mathcal{D}_\lambda)$. Since $\Delta(U_\lambda\,\hat{\otimes}\,E)=\mathcal{D}_\lambda\,\hat{\otimes}\,E$, the free objects as well as the projective objects in $\mathfrak{M}(U_\lambda)$ are mapped into $\mathfrak{M}(\mathcal{D}_\lambda)$ by Δ. Since $\Gamma(\mathcal{M})$ has a well-defined DNF-module structure for each $\mathcal{M}$ in $\mathfrak{M}(\mathcal{D}_\lambda)$, we also have a global sections functor Γ: $\mathfrak{M}(\mathcal{D}_\lambda)\to\mathfrak{M}(U_\lambda)$.

Our objective in this section is to define derived categories associated to $\mathfrak{M}(U_\lambda)$ and $\mathfrak{M}(\mathcal{D}_\lambda)$ and to prove that, for λ regular, the derived functor of Δ gives an equivalence between these two derived categories with inverse functor given by the derived functor of Γ.

We begin with the homotopy categories $K\mathfrak{M}(U_\lambda)$ and $K\mathfrak{M}(\mathcal{D}_\lambda)$ of complexes of DNF U_λ-modules and $\mathcal{D}_\lambda$-modules, respectively, which are cohomologically bounded above, i.e., which have cohomology vanishing for sufficiently high positive degree. We shall also need to consider the subcategories $K^u\mathfrak{M}(U_\lambda)$ and $K^u\mathfrak{M}(\mathcal{D}_\lambda)$ of $K\mathfrak{M}(U_\lambda)$ and $K\mathfrak{M}(\mathcal{D}_\lambda)$ consisting of complexes which are actually bounded from above.

For each integer p we denote by $H^p(M)$ the pth cohomology module of an object M in $K\mathfrak{M}(U_\lambda)$. We emphasize that $H^p(M)$, equipped with the natural quotient topology, need not be a Hausdorff space; we regard it as an object in the category $\mathfrak{M}^0(U_\lambda)$ whose objects are U_λ-modules which also have a (not necessarily Hausdorff) topological vector space structure and whose morphisms are continuous module homomorphisms. If $H^p(M)$ is

Hausdorff, however, then it is also a DNF U_λ-module and belongs to $\mathfrak{M}(U_\lambda)$. We call a morphism f: $M \to N$ in $K\mathfrak{M}(U_\lambda)$ a *quasi-isomorphism* if it induces an algebraic isomorphism on all cohomology modules. Denote by $\Sigma\mathfrak{M}(U_\lambda)$ the class of all quasi-isomorphisms in $K\mathfrak{M}(U_\lambda)$.

In the usual way, using the triangles determined by mapping cones as distinguished triangles (cf. [38; 4, Chap. 1]), $K\mathfrak{M}(U_\lambda)$ is a triangulated category and $\Sigma\mathfrak{M}(U_\lambda)$ is a multiplicative system of morphisms compatible with the triangulated structure. We call the localization of $K\mathfrak{M}(U_\lambda)$ with respect to $\Sigma\mathfrak{M}(U_\lambda)$ *the derived category of* $\mathfrak{M}(U_\lambda)$ and denote it by $D\mathfrak{M}(U_\lambda)$.

According to Corollary A9 in Appendix A, quasi-isomorphisms in $K\mathfrak{M}(U_\lambda)$ are sent to isomorphisms in $\mathfrak{M}^0(U_\lambda)$ by the cohomology functors H^p. Thus, the functors H^p: $K\mathfrak{M}(U_\lambda) \to \mathfrak{M}^0(U_\lambda)$ induce functors $D\mathfrak{M}(U_\lambda) \to \mathfrak{M}^0(U_\lambda)$, which we shall also denote by H^p. In particular it follows that if i^0 denotes the functor which assigns to each module M in $\mathfrak{M}(U_\lambda)$ the class in $D\mathfrak{M}(U_\lambda)$ of the complex which has M in degree zero and $\{0\}$ in every other degree, then i^0 embeds $\mathfrak{M}(U_\lambda)$ as the full subcategory of $D\mathfrak{M}(U_\lambda)$, consisting of complexes with Hausdorff cohomology in degree zero and vanishing cohomology in nonzero degrees. Also, $H^0 \circ i^0$ is naturally isomorphic to the identity.

Exactly the same analysis holds for the category $\mathfrak{M}(\mathscr{D}_\lambda)$. The homotopy category $K\mathfrak{M}(\mathscr{D}_\lambda)$ is a triangulated category and the quasi-isomorphisms form a multiplicative system $\Sigma\mathfrak{M}(\mathscr{D}_\lambda)$ consistent with the triangulated structure. The corresponding localized category we call *the derived category for* $\mathfrak{M}(\mathscr{D}_\lambda)$ and denote by $D\mathfrak{M}(\mathscr{D}_\lambda)$. The pth cohomology of a complex in $K\mathfrak{M}(\mathscr{D}_\lambda)$ may fail to be a DNF sheaf of modules since the topology on its stalks may fail to be Hausdorff. According to Proposition 3.2 the cohomology functors H^p: $K\mathfrak{M}(\mathscr{D}_\lambda) \to \mathfrak{M}^0(\mathscr{D}_\lambda)$ map quasi-isomorphisms to isomorphisms and so we have induced morphisms H^p: $D\mathfrak{M}(\mathscr{D}_\lambda) \to \mathfrak{M}^0(\mathscr{D}_\lambda)$. We have an inclusion i^0: $\mathfrak{M}(\mathscr{D}_\lambda) \to D\mathfrak{M}(\mathscr{D}_\lambda)$, as above, which embeds $\mathfrak{M}(\mathscr{D}_\lambda)$ as a full subcategory of $D\mathfrak{M}(\mathscr{D}_\lambda)$. As before, $H^0 \circ i^0$ is naturally isomorphic to the identity.

It remains to define the derived functors $D\Delta$ and $D\Gamma$. The definition of $D\Gamma$ is quite simple: The Čech resolution of Section 3 provides a functor which assigns to each complex $\mathscr{M}$ in $K\mathfrak{M}(\mathscr{D}_\lambda)$ a complex $\mathscr{C}(\mathscr{M})$ which is still cohomologically bounded above and so belongs to $K\mathfrak{M}(\mathscr{D}_\lambda)$, is quasi-isomorphic to $\mathscr{M}$, and which consists of $\mathscr{D}_\lambda$-fine DNF sheaves. Then $\Gamma(\mathscr{C}(\mathscr{M}))$ is a complex in $K\mathfrak{M}(U_\lambda)$. Clearly, homotopic complexes are sent to homotopic complexes and so $\Gamma \circ \mathscr{C}$ is truly a functor from $K\mathfrak{M}(\mathscr{D}_\lambda)$ to $K\mathfrak{M}(U_\lambda)$. To see that it defines a functor from $D\mathfrak{M}(\mathscr{D}_\lambda)$ to $D\mathfrak{M}(U_\lambda)$, we must show that it maps each quasi-isomorphism ϕ in $K\mathfrak{M}(\mathscr{D}_\lambda)$ to a quasi-isomorphism in $K\mathfrak{M}(U_\lambda)$. By passing to the mapping cone of ϕ and using the long exact sequence for the homology of the associated distinguished triangle, this problem is reduced to showing that $\Gamma \circ \mathscr{C}$ takes exact complexes to exact complexes. However, since Γ is a cohomologically bounded left exact functor (X is a smooth finite dimensional manifold), it preserves exactness of complexes of acyclic sheaves. Thus, $\Gamma \circ \mathscr{C}$ induces a well-defined functor from $D\mathfrak{M}(\mathscr{D}_\lambda)$ to $D\mathfrak{M}(U_\lambda)$, which we shall call the *derived functor $D\Gamma$.*

The derived functor $D\Delta$ is defined similarly but the fact that in some cases Δ may not be homologically bounded forces us to work initially with complexes which are bounded above. The Hochschild resolution (cf. [37, 30]) provides a functor F which assigns to each M in $K^u\mathfrak{M}(U_\lambda)$ a complex $F(M)$ in $K^u\mathfrak{M}(U_\lambda)$ which is quasi-isomorphic to M and which consists of free DNF U_λ-modules. Then $\Delta \circ F$ is a functor from $K^u\mathfrak{M}(U_\lambda)$ to $K^u\mathfrak{M}(\mathscr{D}_\lambda)$. Since F preserves exactness and Δ preserves exactness of complexes of free objects (this is only true for complexes which are bounded above if Δ is not cohomologically bounded), the composition $\Delta \circ F$ maps exact complexes to exact complexes. We now use truncation to extend $\Delta \circ F$ to $K\mathfrak{M}(U_\lambda)$. That is, for an object M in $K\mathfrak{M}(U_\lambda)$, we set

$$(\tau_{\leqslant p} M)^i = \begin{pmatrix} M^i, & i < p \\ \ker d^i, & i = p \\ 0, & i > p \end{pmatrix}.$$

Here d^i is the ith differential map in the complex M (cf. [38; 4, Chap. 1]). Then $\tau_{\leqslant p}$ is a functor from $K\mathfrak{M}(U_\lambda)$ to $K^u\mathfrak{M}(U_\lambda)$ which preserves exactness. Furthermore, for each M in $K\mathfrak{M}(U_\lambda)$, $\tau_{\leqslant p} M$ is quasi-isomorphic to M for sufficiently large p (since M is homologically bounded above). Countable direct limits exist in both $K\mathfrak{M}(U_\lambda)$ and $K\mathfrak{M}(\mathscr{D}_\lambda)$; so we may define a functor from $K\mathfrak{M}(U_\lambda)$ to $K\mathfrak{M}(\mathscr{D}_\lambda)$ by $M \to \lim_p \Delta \circ F \circ \tau_{\leqslant p}(M)$. This preserves exactness and, by the mapping cone argument used above, sends quasi-isomorphisms to quasi-isomorphisms. Hence, it defines a functor, which we shall denote $D\Delta$, from $D\mathfrak{M}(U_\lambda)$ to $D\mathfrak{M}(\mathscr{D}_\lambda)$.

Note that, for λ a regular weight, Δ is a cohomologically bounded functor, as follows, for example, from Propositions 2.4 and 2.5. In this case, $D\Delta$ could be equivalently defined without using truncation: it is easy to see that applying $\Delta \circ F$ to the inclusion $\tau_{\leqslant p} M \to M$ and passing to the limit over p yields a natural isomorphism between the derived functor defined using truncation and the one defined without using it.

To avoid confusion, let us explain our notational convention, relating "cohomology" and "homology." For any complex M, $M_p = M^{-p}$ and $H_p(M) = H^{-p}(M)$. Therefore $\Delta_p(M) = H^{-p}(D\Delta(i^0(M))) = D^{-p}\Delta(i^0(M))$.

We recall that, for a complex M the shifted complex $M[p]$ is defined by $M[p]^k = M^{k+p}$. Set $i^p(M) = i^0(M)[-p]$.

We summarize the above discussion as follows:

5.1. PROPOSITION. (a) *Localizing* $K\mathfrak{M}(U_\lambda)$ *and* $K\mathfrak{M}(\mathscr{D}_\lambda)$ *relative to quasi-isomorphisms yields derived categories* $D\mathfrak{M}(U_\lambda)$ *and* $D\mathfrak{M}(\mathscr{D}_\lambda)$;

(b) *There are well-defined cohomology functors* H^p: $D\mathfrak{M}(U_\lambda) \to \mathfrak{M}^0(U_\lambda)$ *and* H^p: $D\mathfrak{M}(\mathscr{D}_\lambda) \to \mathfrak{M}^0(\mathscr{D}_\lambda)$ *and inclusion functors* i^p: $\mathfrak{M}(U_\lambda) \to D\mathfrak{M}(U_\lambda)$ *and* i^p: $\mathfrak{M}(\mathscr{D}_\lambda) \to D\mathfrak{M}(\mathscr{D}_\lambda)$ *which satisfy* $H^p \circ i^p \cong \mathrm{id}$;

(c) *The functors* Δ *and* Γ *induce derived functors* $D\Delta$: $D\mathfrak{M}(U_\lambda) \to D\mathfrak{M}(\mathscr{D}_\lambda)$ *and* $D\Gamma$: $D\mathfrak{M}(\mathscr{D}_\lambda) \to D\mathfrak{M}(\mathscr{D}_\lambda)$.

Our next objective is to prove that, for regular weight λ, $D\Delta$ and $D\Gamma$ are inverse functors of one another and, hence, define an equivalence between the categories $D\mathfrak{M}(U_\lambda)$ and $D\mathfrak{M}(\mathscr{D}_\lambda)$. One half of this is already quite easy.

5.2. PROPOSITION. *The functor* $D\Gamma \circ D\Delta$ *is naturally isomorphic to the identity regardless of the choise of* λ.

Proof. Tensoring against the identity gives a natural transformation $\mathrm{id} \to \Gamma \circ \Delta$ which is an isomorphism on free modules. In fact if $M = U_\lambda \hat{\otimes} F$ is a free DNF U_λ-module, then $\Delta(M)$ is the free module $\mathscr{D}_\lambda \hat{\otimes} F$, while the higher derived functors Δ_p vanish on M. Furthermore, it follows from Proposition 1.4 and Lemma 3.6 that $\Delta(M)$ is acyclic for the functor Γ and, since Γ has finite cohomological dimension, the natural map $M \to \Gamma \circ \Delta(M)$ is an isomorphism.

Now let M be a complex of free modules which is bounded above. There is a natural quasi-isomorphism $F(M) \to M$ where $F(M)$ is the Hochschild resolution of M. Since both complexes involve modules which are free and, hence, Δ-acyclic, this yields a quasi-isomorphism $\Delta \circ F \to \Delta(M)$. The embedding into the Čech resolution $\Delta \circ F(M) \to \mathscr{C} \circ \Delta \circ F(M)$ is also a quasi-isomorphism. The complexes $\Delta(M)$, $\Delta \circ F(M)$, and $\mathscr{C} \circ \Delta \circ F(M)$ are complexes of sheaves which are acyclic for Γ and so, on applying Γ, we have natural quasi-isomorphisms

$$\Gamma \circ \Delta \circ F(M) \to \Gamma \circ \mathscr{C} \circ \Delta \circ F(M), \qquad \Gamma \circ \Delta \circ F(M) \to \Gamma \circ \Delta(M),$$
$$M \to \Gamma \circ \Delta(M).$$

We conclude that, in the derived category $D\mathfrak{M}(U_\lambda)$, the natural transformation $M \to D\Gamma \circ D\Delta(M)$, is an isomorphism.

5.3. PROPOSITION. *If* λ *is regular, then for every* $\mathscr{D}_\lambda$*-fine sheaf* $\mathscr{F}$ *in* $\mathfrak{M}(\mathscr{D}_\lambda)$, $\Gamma(\mathscr{F})$ *is* Δ*-acyclic and the natural map* $\Delta \circ \Gamma(\mathscr{F}) \to \mathscr{F}$ *is an isomorphism.*

Proof. Let

$$\cdots \to P_1 \to P_0 \to \mathscr{D}_{\lambda,x} \to 0$$

be a resolution of $\mathscr{D}_{\lambda,x}$ by free right DNF U_λ-modules. We know from Theorem 4.1 that $\mathscr{D}_{\lambda,x} \hat{\otimes}_{U_\lambda} \mathscr{F}_x \to \mathscr{F}_x$ is an isomorphism and $\operatorname{Tor}_p^{U_\lambda}(\mathscr{D}_{\lambda,x}, \mathscr{F}_y) = (0)$ if $p > 0$ or $x \neq y$. This implies that the sequence of sheaves

$$\cdots \to P_1 \hat{\otimes}_{U_\lambda} \mathscr{F} \to P_0 \hat{\otimes}_{U_\lambda} \mathscr{F} \to \mathscr{D}_{\lambda,x} \hat{\otimes}_{U_\lambda} \mathscr{F} \to 0$$

is exact and that $\mathscr{D}_{\lambda,x} \hat{\otimes}_{U_\lambda} \mathscr{F}$ is the sheaf supported at $\{x\}$ with stalk $\mathscr{F}_x$. Furthermore, each sheaf in this sequence is fine, since $\mathscr{F}$ is $\mathscr{D}_\lambda$-fine. Thus, the sequence remains exact if we pass to global sections. Since each P_i is free, we have that $P_i \hat{\otimes}_{U_\lambda} \Gamma(\mathscr{F}) \to \Gamma(P_i \hat{\otimes}_{U_\lambda} \mathscr{F})$ is an isomorphism. Thus, we have an exact sequence

$$\cdots \to P_1 \hat{\otimes}_{U_\lambda} \Gamma(\mathscr{F}) \to P_0 \hat{\otimes}_{U_\lambda} \Gamma(\mathscr{F}) \to \mathscr{F}_x \to 0.$$

We have therefore proved that, for each $x \in X$, $\mathscr{D}_{\lambda,x} \hat{\otimes}_{U_\lambda} \Gamma(\mathscr{F}) \to \mathscr{F}_x$ is an isomorphism and $\operatorname{Tor}_p^{U_\lambda}(\mathscr{D}_{\lambda,x}, \Gamma(\mathscr{F})) = (0)$ for $p > 0$. This is equivalent to the statement of the proposition.

5.4. THEOREM. *If λ is regular, then $D\Delta$: $D\mathfrak{M}(U_\lambda) \to D\mathfrak{M}(\mathscr{D}_\lambda)$ is an equivalence of categories with inverse functor $D\Gamma$.*

Proof. In view of Proposition 5.2, we need only show that the natural transformation $\Delta \circ \Gamma \to \mathrm{id}$, given by the module action, induces an isomorphism $D\Delta \circ D\Gamma \to \mathrm{id}$. By the previous proposition, $\Delta \circ \Gamma(\mathscr{F}) \to \mathscr{F}$ is an isomorphism if $\mathscr{F}$ is a $\mathscr{D}_\lambda$-fine DNF sheaf of $\mathscr{D}_\lambda$-modules. By Proposition 3.5, the Čech resolution yields, for any complex in $K\mathfrak{M}(\mathscr{D}_\lambda)$, a natural quasi-isomorphism $\mathscr{F} \to \mathscr{C}(\mathscr{F})$ and $\mathscr{C}(\mathscr{F})$ consists of sheaves which are $\mathscr{D}_\lambda$-fine. Proposition 5.3 now asserts that $\Gamma \circ \mathscr{C}(\mathscr{F})$ consists of Δ-acyclic modules. Thus, the natural quasi-isomorphism $F \circ \Gamma \circ \mathscr{C}(\mathscr{F}) \to \Gamma \circ \mathscr{C}(\mathscr{F})$, given by the Hochschild resolution, is preserved on applying Δ. Consequently, we have natural quasi-isomorphisms

$$\Delta \circ F \circ \Gamma \circ \mathscr{C}(\mathscr{F}) \to \Delta \circ \Gamma \circ \mathscr{C}(\mathscr{F}) \to \mathscr{C}(\mathscr{F}) \qquad \text{and} \qquad \mathscr{F} \to \mathscr{C}(\mathscr{F}).$$

This proves that, in the derived category $D\mathfrak{M}(\mathscr{D}_\lambda)$, there is a natural isomorphism $D\Delta \circ D\Gamma \to \mathrm{id}$.

6. INTERTWINING FUNCTORS

According to the results in Section 5, if λ is regular the functors $D\Gamma$: $D\mathfrak{M}(\mathscr{D}_\lambda) \to D\mathfrak{M}(U_\lambda)$ and $D\Delta$: $D\mathfrak{M}(U_\lambda) \to D\mathfrak{M}(\mathscr{D}_\lambda)$ are equivalences of categories, and inverses of each other. Of course, if w is a Weyl group element, then U_λ and $U_{w\lambda}$ are naturally isomorphic as algebras (though not as $U(\mathfrak{h})$-algebras) and so there is a natural equivalence

J_w: $D\mathfrak{M}(U_\lambda) \to D\mathfrak{M}(U_{w\lambda})$, which leads, through composition, to an equivalence of categories $D\Delta \circ J_w \circ D\Gamma$: $D\mathfrak{M}(\mathscr{D}_\lambda) \to D\mathfrak{M}(\mathscr{D}_{w\lambda})$. As written, this functor does not seem to be very interesting. The point is, however, that for certain w's it is equivalent to a much more concrete functor that can be used in explicit calculations. We will be interested in the case when $w = s(\alpha)$ for a simple root α.

Our discussion of the intertwining functor will depend on the study of the projection π: $X \to Y$, where Y is the variety of parabolics of type α. However, much of the machinery we shall develop in this connection will also be needed in a more general situation—that in which Y is the variety of parabolic subalgebras of $\mathfrak{g}$ of an arbitrary given conjugacy class. Thus, we shall initially work in this setting and later specialize to the case where Y is the variety of parabolics of type α.

Given a conjugacy class Y of parabolic subalgebras of $\mathfrak{g}$, each Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is contained in a unique parabolic subalgebra $\mathfrak{p}$ in Y (cf. [5]). The space Y is naturally a G-homogeneous complex projective variety. The projection π: $X \to Y$ which assigns $\mathfrak{p}$ to $\mathfrak{b}$ is G-invariant. Fix $\mathfrak{b}$ and $\mathfrak{p}$ as above. Let B and P be analytic subgroups of G with Lie algebras $\mathfrak{b}$ and $\mathfrak{p}$, respectively. With these choices made we can identify X with G/B and Y with G/P. Also, π becomes the natural projection $G/B \to G/P$. The fiber of π over $\mathfrak{p}$ is P/B which is just the flag variety of the "Levi factor" $\mathfrak{l}$ which is $\mathfrak{p}$ modulo its nilpotent radical. We remark that $\mathfrak{l}$ is a reductive Lie algebra. As in the case of the flag variety, we will often denote the points of Y by lower case roman letters. For $y \in Y$, $\mathfrak{p}_y$ denotes the corresponding parabolic subalgebra.

Since π is a proper projection, $\pi_*(\mathscr{D}_\lambda)$ is a DNF sheaf of algebras on Y. We need to have an explicit description of the sheaf $\pi_*(\mathscr{D}_\lambda)$. Let $\mathfrak{u}_y$ be the nilpotent radical of $\mathfrak{p}_y$ and $\mathfrak{l}_y = \mathfrak{p}_y/\mathfrak{u}_y$ the "Levi factor" of $\mathfrak{p}_y$. The collection $\{\mathfrak{p}_y\}$ (resp. $\{\mathfrak{u}_y\}$, $\{\mathfrak{l}_y\}$) forms a G-homogeneous holomorphic vector bundle on Y. We denote by $\mathscr{P}_Y$ (resp. $\mathscr{U}_Y$, $\mathscr{L}_Y$) its sheaf of local holomorphic sections. These sheaves give rise to sheaves of universal enveloping algebras $U(\mathscr{P}_Y)$, $U(\mathscr{U}_Y)$, and $U(\mathscr{L}_Y)$. For example, $U(\mathscr{P}_Y)$ is a sheaf of local sections of the bundle with fiber $U(\mathfrak{p}_y)$ over y. Similarly we define the sheaf $\mathscr{G}_Y = \mathcal{O}_Y \otimes \mathfrak{g}$ of local sections of the trivial bundle with fiber $\mathfrak{g}$, and by $U(\mathscr{G}_Y)$, the sheaf $O_Y \otimes U(\mathfrak{g})$. We note that $U(\mathscr{G}_Y)$ has a natural structure of a sheaf of algebras: the definition of the product is completely analogous to that on $U(\mathscr{G}_X) = U(\mathscr{G})$ given in Section 1. The G-homogeneous subsheaf $Z(\mathscr{L}_Y)$ of $U(\mathscr{L}_Y)$ consisting of sections admitting values in $Z(\mathfrak{l}_y)$ at y, for each y, is free. In fact, the adjoint representation of the isotropy group P_y of y on $Z(\mathfrak{l}_y)$ is trivial. We denote the space of constant sections of $Z(\mathscr{L}_Y)$ by $Z(\mathfrak{l})$. Thus, although we cannot talk about an abstract Levi factor, the notion of an abstract center of the universal enveloping algebra of the Levi factor still makes sense.

Since $\mathscr{U}_Y$ is a sheaf ideals in $\mathscr{G}_Y$, $\mathscr{U}_Y U(\mathscr{G}_Y)$ is a two-sided ideal in $U(\mathscr{G}_Y)$. Let $\mathscr{D}_{\mathfrak{l}}$ denote the quotient algebra. There is a natural isomorphisms of $Z(\mathfrak{l})$ onto the center of $\mathscr{D}_{\mathfrak{l}}$. There is also a natural homomorphism $Z(\mathfrak{l}) \to U(\mathfrak{h})$ defined as follows. Fix y in Y. The "evaluation at y" map identifies $Z(\mathfrak{l})$ with $Z(\mathfrak{l}_y)$. Similarly, the "restriction to $\pi^{-1}(y)$" map identifies $\mathfrak{h}$ with the abstract Cartan subalgebra of $\mathfrak{l}_y$. The resulting unnormalized Harish-Chandra homomorphism $Z(\mathfrak{l}) \to U(\mathfrak{h})$ is independent of the choice of y.

We are now ready to describe the sheaf $\pi_*(\mathscr{D}_{\mathfrak{h}})$. First we observe that the natural homomorphism of $U(\mathscr{G}_Y)$ into $\pi_*(\mathscr{D}_{\mathfrak{h}})$ lifts to $\mathscr{D}_{\mathfrak{l}}$. This follows from the fact that $\mathfrak{u}_y$ is contained in $\mathfrak{n}_x$ for each x in $\pi^{-1}(y)$. We have also a natural homomorphism $\mathfrak{h} \to \pi_*(\mathscr{D}_{\mathfrak{h}})$. Together they induce an algebra homomorphism

$$\mathscr{D}_{\mathfrak{l}} \otimes_{Z(\mathfrak{l})} U(\mathfrak{h}) \to \pi_*(\mathscr{D}_{\mathfrak{h}}). \tag{6.1a}$$

Now, let λ be an element of $\mathfrak{h}^*$. Then (6.1a) induces a sequence of morphisms:

$$\begin{aligned} \mathscr{D}_{\mathfrak{l}} \otimes_{Z(\mathfrak{l})} \mathbb{C}_{\lambda+\rho} &= \mathscr{D}_{\mathfrak{l}} \otimes_{Z(\mathfrak{l})} U(\mathfrak{h}) \otimes_{U(\mathfrak{h})} \mathbb{C}_{\lambda+\rho} \\ &\to \pi_*(\mathscr{D}_{\mathfrak{h}}) \otimes_{U(\mathfrak{h})} \mathbb{C}_{\lambda+\rho} \\ &\to \pi_*(\mathscr{D}_{\mathfrak{h}} \otimes_{U(\mathfrak{h})} \mathbb{C}_{\lambda+\rho}) = \pi_*(\mathscr{D}_\lambda), \end{aligned}$$

and in particular the composite morphism

$$\mathscr{D}_{\mathfrak{l}} \otimes_{Z(\mathfrak{l})} \mathbb{C}_{\lambda+\rho} \to \pi_*(\mathscr{D}_\lambda). \tag{6.1b}$$

6.1. PROPOSITION. *The maps* (6.1a), (6.1b) *are isomorphisms of algebras. Moreover,* $\mathscr{D}_{\mathfrak{h}}$ *and* $\mathscr{D}_\lambda$ *are acyclic for* π_*.

Part (b) of this proposition is proved in [12, Proposition 4.5].

Proof. We note that all the sheaves of algebras in question are filtered by locally free $\mathcal{O}_Y$-modules of finite rank (the filtration being induced by that of $U(\mathfrak{g})$) and the maps in (6.1) preserve this filtration. Thus it is enough to prove that the induced maps on geometric fibers are isomorphisms. Fix $y \in Y$ and write $\mathfrak{n}$, $\mathfrak{p}$, $\mathfrak{l}$ instead of $\mathfrak{n}_y$, $\mathfrak{p}_y$, and $\mathfrak{l}_y$. The geometric fiber of $\mathscr{D}_{\mathfrak{l}}$ at y is isomorphic, as a right $U(\mathfrak{g})$-module, to $\mathbb{C} \otimes_{U(\mathfrak{u})} U(\mathfrak{g})$, or, equivalently, to $U(\mathfrak{l}) \otimes_{U(\mathfrak{p})} U(\mathfrak{g})$. Consequently, the geometric fiber of $\mathscr{D}_{\mathfrak{l}} \otimes_{Z(\mathfrak{l})} U(\mathfrak{h})$ is isomorphic to $(U(\mathfrak{l}) \otimes_{Z(\mathfrak{l})} U(\mathfrak{h})) \otimes_{U(\mathfrak{p})} U(\mathfrak{g})$. Here $\mathfrak{h}$ is regarded as the abstract Cartan subalgebra of $\mathfrak{l}$. Similarly, the geometric fiber of $\mathscr{D}_{\mathfrak{l}} \otimes_{Z(\mathfrak{l})} \mathbb{C}_{\lambda+\rho}$ at y is isomorphic to $(U(\mathfrak{l}) \otimes_{Z(\mathfrak{l})} \mathbb{C}_{\lambda+\rho}) \otimes_{U(\mathfrak{p})} U(\mathfrak{g})$.

Now, let j: $\pi^{-1}(y) \to X$, i: $\{y\} \to Y$ be the inclusions, and π': $\pi^{-1}(y) \to \{y\}$

the projection. The "geometric fiber" $j^*(\mathscr{D}_{\mathfrak{h}})$ of $\mathscr{D}_{\mathfrak{h}}$ along $\pi^{-1}(y)$ is naturally isomorphic to $\mathscr{D}_{\mathfrak{l},\mathfrak{h}} \otimes_{U(\mathfrak{p})} U(\mathfrak{g})$, where $\mathscr{D}_{\mathfrak{l},\mathfrak{h}}$ is the analogue of $\mathscr{D}_{\mathfrak{h}}$ with $\mathfrak{l}$ playing the role of $\mathfrak{g}$. Similarly, $j^*(\mathscr{D}_\lambda) \cong (\mathscr{D}_{\mathfrak{l},\mathfrak{h}} \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda+\rho}) \otimes_{U(\mathfrak{p})} U(\mathfrak{g}) = (\mathscr{D}_{\mathfrak{l},[\lambda+\rho]}) \otimes_{U(\mathfrak{p})} U(\mathfrak{g})$, where $\mathscr{D}_{\mathfrak{l},[\lambda+\rho]}$ is the analogue of $\mathscr{D}_{[\mu]}$ (cf. (1.2)) with $\mathfrak{l}$ playing the role of $\mathfrak{g}$. Since π is a proper projection, and $\mathscr{D}_{\mathfrak{h}}$ and $\mathscr{D}_\lambda$ are locally free sheaves, we have

$$\begin{aligned} i^*D^k\pi_*(\mathscr{D}_{\mathfrak{h}}) &\cong D^k\pi_*j^*(\mathscr{D}_{\mathfrak{h}}) \cong D^k\pi'_*(\mathscr{D}_{\mathfrak{l},\mathfrak{h}}) \otimes_{U(\mathfrak{p})} U(\mathfrak{g}) \\ i^*D^k\pi_*(\mathscr{D}_\lambda) &\cong D^k\pi_*j^*(\mathscr{D}_\lambda) \cong D^k\pi'_*(\mathscr{D}_{\mathfrak{l},[\lambda+\rho]} \otimes_{U(\mathfrak{p})} U(\mathfrak{g}), \end{aligned} \tag{6.2}$$

where $D^k\pi_*$ is the kth derived functor of π_*. It follows from Proposition 1.2 and Proposition 1.4 applied to the case $\mathfrak{g}=\mathfrak{l}$ that both $\mathscr{D}_{\mathfrak{h}}$ and $\mathscr{D}_\lambda$ are π_*-acyclic. With the above identifications, the maps induced by (6.1) on the geometric fiber at y,

$$\begin{aligned} (U(\mathfrak{l}) \otimes_{Z(\mathfrak{l})} U(\mathfrak{h})) \otimes_{U(\mathfrak{p})} U(\mathfrak{g}) &\to \Gamma(\pi^{-1}(y), \mathscr{D}_{\mathfrak{l},\mathfrak{h}}) \otimes_{U(\mathfrak{p})} U(\mathfrak{g}) \\ (U(\mathfrak{l}) \otimes_{Z(\mathfrak{l})} \mathbb{C}_{\lambda+\rho}) \otimes_{U(\mathfrak{p})} U(\mathfrak{g}) &\to \Gamma(\pi^{-1}(y), \mathscr{D}_{\mathfrak{l},[\lambda+\rho]}) \otimes_{U(\mathfrak{p})} U(\mathfrak{g}), \end{aligned} \tag{6.3}$$

are obtained from the natural inclusion maps

$$\begin{aligned} (U(\mathfrak{l}) \otimes_{Z(\mathfrak{l})} U(\mathfrak{h})) &\to \Gamma(\pi^{-1}(y), \mathscr{D}_{\mathfrak{l},\mathfrak{h}}) \\ (U(\mathfrak{l}) \otimes_{Z(\mathfrak{l})} \mathbb{C}_{\lambda+\rho}) &\to \Gamma(\pi^{-1}(y), \mathscr{D}_{\mathfrak{l},[\lambda+\rho]}) \end{aligned}$$

by tensoring with $U(\mathfrak{g})$. Appealing again to Propositions 1.2 and 1.4 we conclude that those maps are isomorphisms. This completes the proof of the proposition.

The identification of the abstract Catan subalgebra of $\mathfrak{l}_y$ with the abstract Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ identifies the root system of $\mathfrak{l}_y$ with a subset of the root system of $\mathfrak{g}$. This subset is independent of y and we denote it by $\Delta(\mathfrak{l})$. The set $\Delta^+(\mathfrak{l})$ of positive roots in $\Delta(\mathfrak{l})$ is also defined independent of y and is identified with a subset of the set Δ^+ of positive roots in Δ. The Weyl group $W(\mathfrak{l})$ of $\mathfrak{l}_y$ may be identified with the subgroup of the Weyl group W of $\mathfrak{g}$ which leaves invariant the set of roots corresponding to $\mathfrak{u}_y$. As a group of transformations of $\mathfrak{h}$, it is also independent of y.

6.2. Corollary (cf. [12, Lemma 4.4]). *If $w \in W(\mathfrak{l})$, then $\pi_*(\mathscr{D}_\lambda)$ and $\pi_*(\mathscr{D}_{w\lambda})$ are naturally isomorphic as sheaves of algebras.*

Proof. In fact, as $U(\mathfrak{h})$-modules, $U(\mathfrak{h}) \otimes_{Z(\mathfrak{l})} \mathbb{C}_{\lambda+\rho} = U(\mathfrak{h}) \otimes_{Z(\mathfrak{l})} \mathbb{C}_{w\lambda+\rho}$, which implies that $\pi_*(\mathscr{D}_{\mathfrak{l}} \otimes_{Z(\mathfrak{l})} \mathbb{C}_{\lambda+\rho}) \cong \pi_*(\mathscr{D}_{\mathfrak{l}} \otimes_{Z(\mathfrak{l})} \mathbb{C}_{w\lambda+\rho})$.

We denote the sheaf of algebras $\pi_*(\mathscr{D}_\lambda)$ by $\mathscr{A}_\lambda(Y)$. For each variety of parabolics Y, this is a sheaf of algebras on Y. The extreme cases are when

$Y = X$ and when $Y = \{\text{pt.}\}$ (where pt. $= G/G$). We have $\mathscr{A}_\lambda(X) = \mathscr{D}_\lambda$, while $\mathscr{A}_\lambda(\text{pt.}) = U_\lambda(\mathfrak{g})$. For each G-covariant morphism π: $Y_1 \to Y_2$ between such varieties there is a natural isomorphism $\pi_* \mathscr{A}_\lambda(Y_1) \to \mathscr{A}_\lambda(Y_2)$.

For a given variety of parabolics Y, let $\mathfrak{M}(\mathscr{A}_\lambda(Y))$ denote the category of DNF $\mathscr{A}_\lambda(Y)$-modules. We construct its derived category $D\mathfrak{M}(\mathscr{A}_\lambda(Y))$ in precisely the same manner as we constructed $D\mathfrak{M}(\mathscr{D}_\lambda)$. Let π: $X \to Y$ be the projection. For $\mathscr{F}$ in $\mathfrak{M}(\mathscr{A}_\lambda(Y))$ define

$$\Delta_\lambda^Y(\mathscr{F}) = \mathscr{D}_\lambda \mathbin{\hat{\otimes}}_{\pi^{-1}(\mathscr{A}_\lambda(Y))} \pi^{-1}(\mathscr{F}).$$

We note that $\Delta_\lambda^Y(\mathscr{F})$ is a $\mathscr{D}_\lambda$-module, which is not DNF, in general, due to the possibility that its stalks may not be Hausdorff. As in the case of the localization functor Δ, we will construct the derived functor of Δ_λ^Y.

A *quasifree* $\mathscr{A}_\lambda(Y)$*-module* is a sheaf of the form $\mathscr{A}_\lambda(Y) \mathbin{\hat{\otimes}} \mathscr{H}$, where $\mathscr{H}$ is a DNF sheaf of topological vector spaces on Y. We note that a quasifree $\mathscr{A}_\lambda(Y)$-module is in $\mathfrak{M}(\mathscr{A}_\lambda(Y))$, and that every free module is quasifree (a free module has the same form, but with $\mathscr{H}$ a fixed topological vector space rather than a sheaf). We also note that every complex $\mathscr{F}$ consisting of modules in $\mathfrak{M}(\mathscr{A}_\lambda(Y))$ has a canonical resolution $F(\mathscr{F}) \to \mathscr{F}$ consisting of quasifree modules—namely, the Hochschild resolution. We define

$$D\Delta_\lambda^Y(\mathscr{F}) = \Delta_\lambda^Y(F(\mathscr{F})).$$

This defines a functor $D\Delta_\lambda^Y$: $D\mathfrak{M}(\mathscr{A}_\lambda(Y)) \to D\mathfrak{M}(\mathscr{D}_\lambda))$. We observe that $F(\mathscr{F})$ can be replaced by any quasifree resolution of $\mathscr{F}$.

We also define the derived functor of π_*

$$D\pi_*\colon D\mathfrak{M}(\mathscr{D}_\lambda) \to D\mathfrak{M}(\mathscr{A}_\lambda(Y))$$

by $D\pi_*(\mathscr{F}) = \pi_*(\mathscr{C}(\mathscr{F}))$, where, as in Section 5, $\mathscr{C}(\mathscr{F})$ denotes the Čech resolution of $\mathscr{F}$. Again, we can replace $\mathscr{C}(\mathscr{F})$ by any π_*-acyclic resolution of $\mathscr{F}$.

The functors $D^p\Delta_\lambda^Y(\mathscr{H}) = H^p(D\Delta_\lambda^Y(\mathscr{H}))$ for $\mathscr{H}$ in $D\mathfrak{M}(\mathscr{A}_\lambda(Y))$ can be computed using a Koszul resolution analogous to that in Section 1. First let us consider the sheaf $\pi^*(\mathscr{U}_Y)$ of ideals of $\mathscr{N}$, and form the quotient $\mathscr{N}_\mathrm{u} = \mathscr{N}/\pi^*(\mathscr{U}_Y)$. Since $\pi^*(\mathscr{U}_Y)$ forms a sheaf of ideals in $\mathscr{G}$, $\pi^*(\mathscr{U}_Y)\, U(\mathscr{G})$ is a two-sided ideal in $U(\mathscr{G})$. The quotient, $U(\mathscr{G})/\pi^*(\mathscr{U}_Y)\, U(\mathscr{G})$ is naturally isomorphic to $\pi^*(\mathscr{D}_\mathfrak{l})$. Moreover, there is a resolution

$$K(\mathscr{N}_\mathrm{u}, \mathscr{A}_\lambda(Y)) \to \mathscr{D}_\mathfrak{h} \to 0$$

which is a relative version of the Koszul resolution $K(\mathscr{N}, \mathscr{D}_\mathfrak{h}) \to \mathscr{D}_\mathfrak{h}$ of Section 1. Here, for each $\mathscr{H}$ in $\mathfrak{M}(\mathscr{A}_\lambda(Y))$ we denote by $K(\mathscr{N}_\mathrm{u}, \mathscr{H})$ the Koszul complex determined by the action of the sheaf of Lie algebras $\mathscr{N}_\mathrm{u}$

on the sheaf of modules $\pi^*(\mathscr{H})$. The differentials in this complex commute with the natural left $U(\mathscr{G})$-action. In particular, the homology groups

$$H_k(\mathscr{N}_u, \mathscr{H}) = H_k(K(\mathscr{N}_u, \mathscr{H}))$$

have a natural $U(\mathfrak{h})$-module structure for every $\mathscr{H}$ in $\mathfrak{M}(\mathscr{A}_\lambda(Y))$. Moreover $H_k(\mathscr{N}_u, \mathscr{H})$ splits into a direct sum of weight subspaces corresponding to $w\lambda + \rho$ where w runs through the Weyl group for $\mathfrak{l}$. The proof of the following proposition is analogous to the proof of Proposition 2.4.

6.3. PROPOSITION. *For each* $\mathscr{H} \in \mathfrak{M}(\mathscr{A}_\lambda(Y))$, $D_k \Delta_\lambda^Y(\mathscr{H})$ *is naturally isomorphic to the* $(\lambda + \rho)$*-weight subspace of* $H_k(\mathscr{N}_u, \mathscr{H})$.

We now specialize to the case where Y is the variety of parabolics of type α for a simple root $\alpha \in \Delta^+$. A typical such parabolic $\mathfrak{p}_y$ is spanned by a Borel algebra $\mathfrak{b}_x$ (with $\pi(x) = y$) and the root space corresponding to the root $-\alpha_x$, where α_x is the specialization of α to the point x. Here, the fibers $\pi^{-1}(y)$ are copies of P^1; the Levi factors $\mathfrak{l}_y$ have as semisimple part, $[\mathfrak{l}_y, \mathfrak{l}_y]$, copies of $\mathfrak{sl}_2(\mathbb{C})$; and the sheaf of Lie algebras $\mathscr{N}_u = \mathscr{N}/\pi^*(\mathscr{U}_Y)$ may be identified with the sheaf of local sections of the line bundle whose fiber at x is the root space of α.

By Corollary 6.2, there is a natural isomorphism between $\mathscr{A}_\lambda(Y)$ and $\mathscr{A}_{s(\alpha)\lambda}(Y)$ and, hence, an equivalence of categories $\mathfrak{M}(\mathscr{A}_\lambda(Y)) \to \mathfrak{M}(\mathscr{A}_{s(\alpha)\lambda}(Y))$ between the corresponding categories of modules. We let Δ_α denote the composition of this functor with the localization functor $\Delta^Y_{s(\alpha)\lambda}$: $\mathfrak{M}(\mathscr{A}_{s(\alpha)\lambda}(Y)) \to \mathfrak{M}(\mathscr{D}_{s(\alpha)\lambda})$. We then define the *intertwining functor* $\mathscr{T}_\alpha$: $D\mathfrak{M}(\mathscr{D}_\lambda) \to D\mathfrak{M}(\mathscr{D}_{s(\alpha)\lambda})$ by $\mathscr{T}_\alpha = D\Delta_\alpha \circ D\pi_*$.

6.4. PROPOSITION. *As functors from* $D\mathfrak{M}(U_\lambda)$ *to* $D\mathfrak{M}(\mathscr{D}_{s(\alpha)\lambda})$, $\mathscr{T}_\alpha \circ D\Delta$ *and* $D\Delta \circ J_{s(\alpha)}$ *are naturally isomorphic.*

Proof. Let F be a free resolution of M. Then each F^k is a free DNF U_λ-module, and hence it can be written as $U_\lambda \hat{\otimes} N^k$, where N^k is a DNF topological vector space. Thus, as a graded module, $\Delta F \cong \mathscr{D}_\lambda \hat{\otimes} N$, where $N = \bigoplus N^k$. By Proposition 6.1, $\mathscr{D}_\lambda \hat{\otimes} N$ is π_*-acyclic and so $D\pi_*(\mathscr{D}_\lambda \hat{\otimes} N) \cong \pi_*(\mathscr{D}_\lambda \hat{\otimes} N)$. Also, $\pi_*(\mathscr{D}_\lambda \hat{\otimes} N) \cong \pi_*(\mathscr{D}_\lambda) \hat{\otimes} N$. Thus, $\mathscr{T}_\alpha \circ D\Delta(M) \cong D\Delta_\alpha(\pi_*(\mathscr{D}_\lambda) \hat{\otimes} N) \cong \Delta_\alpha(\pi_*(\mathscr{D}_\lambda) \hat{\otimes} N)$, as the complex $\pi_*(\mathscr{D}_\lambda) \hat{\otimes} N$ is Δ_α-acyclic. Now, $\Delta_\alpha(\pi_*(\mathscr{D}_\lambda) \hat{\otimes} N) \cong \Delta_\alpha(\pi_*(\mathscr{D}_\lambda)) \hat{\otimes} N \cong \mathscr{D}_{s(\alpha)\lambda} \hat{\otimes} N \cong \Delta \circ J_{s(\alpha)}(F) \cong D\Delta \circ J_{s(\alpha)}(M)$. This completes the proof of the proposition.

As an immediate consequence of Proposition 6.4 and Theorem 5.4 we get the following corollary.

6.5. COROLLARY. *The intertwining functor* $\mathscr{T}_\alpha\colon D\mathfrak{M}(\mathscr{D}_\lambda) \to D\mathfrak{M}(\mathscr{D}_{s(\alpha)\lambda})$ *is an equivalence of categories.*

We will need to have tools to compute the cohomology functors $\mathscr{T}_\alpha^k \cong H^k\mathscr{T}_\alpha$ of $\mathscr{T}_\alpha$. Let $\mathscr{F}$ be a complex in $D\mathfrak{M}(\mathscr{D}_\lambda)$. By Theorem 5.4, $\mathscr{F}$ can be resolved by a complex $\mathscr{F}^{\sim}$ consisting of free $\mathscr{D}_\lambda$-modules. By Proposition 6.1, $D\pi_*(\mathscr{F}^{\sim}) \cong \pi_*(\mathscr{F}^{\sim})$, which in turn is a complex acyclic for $D\Delta_\alpha$. Let $\mathscr{K}$ be a complex in $D\mathfrak{M}(\pi_*(\mathscr{D}_\lambda))$. The Hochschild resolution $F(\mathscr{K})$ of $\mathscr{K}$ is a double complex, with horizontal and vertical differentials d_{I} and d_{II}, where d_{I} comes from the differential in $\mathscr{K}$ and d_{II} from the Hochschild resolution. If the cohomology sheaves of d_{I} are DNF-sheaves, $H_{\mathrm{I}}^{k,*}(F(\mathscr{K}))$ becomes in a natural way the Hochschild resolution of $H^k(\mathscr{K})$. Thus the conditions for the existence of a Grothendieck spectral sequence apply in this situation (cf. [15, Chap. II, Sect. 4]). When we apply this to $\mathscr{K} = \pi_*(\mathscr{F}^{\sim})$ we have the result:

6.6. PROPOSITION. *Let* $\mathscr{F}$ *be a complex in* $D\mathfrak{M}(\mathscr{D}_\lambda)$. *Assume that for each* k, $D^k\pi_*(\mathscr{F})$ *is a DNF-sheaf. Then there exists a spectral sequence*

$$D^p\,\Delta_\alpha(D^q\pi_*(\mathscr{F})) \Rightarrow \mathscr{T}_\alpha^{p+q}(\mathscr{F}).$$

It is clear that with obvious modifications we can establish analogous facts for a variety of parabolics of an arbitrary type.

7. ANALYTIC MODULES

Here we shall discuss the class of analytic $U(\mathfrak{g})$-modules and state some key results on globalization of $U(\mathfrak{g})$-modules due to Schmid [33].

Roughly speaking, an analytic $U(\mathfrak{g})$-module is a $U(\mathfrak{g})$-module M such that, for each $m \in M$, the action of $\mathfrak{g}$ on m exponentiates to a complex analytic action of a neighborhood of the identity in G. More specifically: Let M be a topological $U(\mathfrak{g})$-module and let U be an open set in G. Then there are three natural left $\mathfrak{g}$-module structures on the space $O(U) \hat{\otimes} M$, of holomorphic M-valued functions on U. These are defined by letting $\xi \in \mathfrak{g}$ act as $1 \hat{\otimes} \xi$, $\xi_r \hat{\otimes} 1$, and $\xi_l \hat{\otimes} 1$, where ξ_r (resp. $-\xi_l$) is the left (resp. right) invariant vector field determined by ξ.

If $f \in \mathcal{O}(U) \hat{\otimes} M$ satisfies $(\xi_l \hat{\otimes} 1)f + (1 \hat{\otimes} \xi)f = 0$ for all $\xi \in \mathfrak{g}$, then we will say f is *exponential*. Clearly, a function which is exponential has its entire power series expansion at a point $g \in U$ determined by its value at g. Thus, such a function is determined on a connected set by its value at any point of this set.

7.1. DEFINITION. Let M be a topological $U(\mathfrak{g})$-module. An element $m \in M$ will be called *analytic* if there is a neighborhood U of the identity e in G and an exponential function $f \in \mathcal{O}(U) \hat{\otimes} M$ such that $f(e) = m$. If every element of M is analytic then M will be called an *analytic* $U(\mathfrak{g})$-*module.*

Note that a right translate of an exponential function is also an exponential function. It follows that the values of an exponential function on any domain are all analytic vectors. For the same reason, any left invariant vector field, applied to an exponential function is another exponential function. Since $(\xi_r \hat{\otimes} 1) f$ and $-(\xi_l \hat{\otimes} 1) f$ agree at the identity of G and have the value $(1 \hat{\otimes} \xi) f(e)$ if f is exponential, we conclude that if m is the value at e of an exponential function on U, then so is ξm for all $\xi \in \mathfrak{g}$. Thus, the set of analytic vectors in a given $\mathfrak{g}$-module forms a submodule.

If G_0 is a real Lie group with complexified Lie algebra $\mathfrak{g}$, (ω, V_ω) is a representation of G_0, and $v \in V_\omega$ is an analytic vector in the sense that $g \to \omega(g)v$ is a real analytic function, then a holomorphic extension of this function to a neighborhood of e in G is an example of an exponential function. Thus, in this setting, a vector in V_ω is an analytic vector in our sense if and only if it is an analytic vector in the usual sense.

Suppose K is a real or complex Lie group, together with a homomorphism $\varphi: K \to G$ with the following property: the differential of φ is injective, and the subgroup $\varphi(K)$ is closed in G. In particular the Lie algebra $\mathfrak{k}$ of K can be regarded as a subalgebra of $\mathfrak{g}$. Let M be a $(\mathfrak{g}, K)$-*module*; that is, there are representations of K and $\mathfrak{g}$ on M (both denoted by ω) which satisfy the following compatibility conditions:

(1) the action of $\mathfrak{k}$, regarded as a subalgebra of $\mathfrak{g}$, coincides with the differential of the action of K; (7.1)

(2) for $k \in K$ and $\xi \in \mathfrak{g}$, $\omega(k)\omega(\xi)\omega(k^{-1}) = \omega(\mathrm{Ad}(k)(\xi))$.

Suppose M is also analytic. For $m \in M$ we let f be an exponential function from a neighborhood U of e in G to M with $f(e) = m$. Then the function $h: K \times U \to M$ defined by $h(k, g) = \omega(k) f(g)$ satisfies the identity $h(kk_1, g) = h(k, \varphi(k_1)\, g)$ for k_1 in a neighborhood of e in K. This is because h is analytic in its second variable and the identity is satisfied infinitesimally due to (7.1) above and the definition of exponential function. It follows that h yields a well-defined global section of the sheaf $\mathcal{O}_{G|K} \hat{\otimes} M$ on K, where $\mathcal{O}_{G|K}$ *is defined to be the pullback*, $\varphi^{-1}\mathcal{O}_G$, *of* $\mathcal{O}_G$ *to* K. If K happens to be a subgroup of G, $\mathcal{O}_{G|K}$ is the restriction of $\mathcal{O}_G$ to K and h is naturally an extension of f to an exponential function in a neighborhood of K in G, which agrees with $k \to \omega(k)m$ on K.

It is convenient to think of elements of $\varphi^{-1}\mathcal{O}_G$ as germs of functions. Let

V be a neighborhood of 0 in $\mathfrak{g}$ which is biholomorphic, under the exponential map, to a neighborhood of e in G. Define $G|K(V)$ as the quotient of $K \times (\exp V)$ by the relation $(kk', g) = (k, \varphi(k')\, g)$ for k' in $\exp_K(V \cap \mathfrak{k})$ and g such that both g and $\varphi(k')\, g$ are in $\exp V$. We note that K is a closed submanifold of $G|K(V)$ and φ extends to a map $G|K(V) \to G$ defined by $\varphi(k, g) = \varphi(k)\, g$. If V is small enough φ is a local homeomorphism. Define $G|K$ to be the limit in the sense of germs of sets of the sets $G|K(V)$. We will call this the infinitesimal neighborhood of K in G. Now, $\mathcal{O}_{G|K}$ can be regarded as a sheaf of germs of holomorphic functions on $G|K$. At a given $k \in K$ a germ in $(\mathcal{O}_{G|K})_k$ can be thought of as a germ f_k of a holomorphic function defined in a neighborhood of $\varphi(k)$ in G. The dependence on K is not important locally, since $f_{kk'}(g) = f_k(\varphi(k')\, g)$. For example, we denote the above element $h(k, g)$ by $\omega_k(g)m$, but we shall often drop the "k" dependence and denote this simply by $\omega(g)m$, where g stands for an element of a representative set of $G|K$. We call this function *the exponential function for m on $G|K$*.

We use the correspondence $m \to \omega(\text{-})m$ to define an $\mathcal{O}_{G|K}$-module homomorphism

$$\theta\colon \mathcal{O}_{G|K} \hat{\otimes} M \to \mathcal{O}_{G|K} \hat{\otimes} M$$

by $\theta(f \hat{\otimes} m)(g) = f(g)\omega(g)m$. Note that θ is characterized by the fact that it is an $\mathcal{O}_{G|K}$-module homomorphism for which $\theta(1 \hat{\otimes} m)$ is an exponential function in $\mathcal{O}_{G|K}$ which agrees with $k \to \omega(k)m$ on K.

Our goal is to give a criterion, in terms of the homomorphism $\theta\colon \mathcal{O}_{G|K} \hat{\otimes} M \to \mathcal{O}_{G|K} \hat{\otimes} M$, for a DNF module M to have a structure of an analytic $(\mathfrak{g}, K)$-module.

The associative law, $\omega(g_1 g_2)m = \omega(g_1)\omega(g_2)m$, holds as a relation between germs of function on $G|K \times G|K$. To prove this we note first that the relation $\omega(k_1 g_2)m = \omega(k_1)\omega(g_2)m$ for $k_1 \in K$ and g_2 in a representative of $G|K$ follows from the associativity of the K action and the above construction of the extension of $k \to \omega(k)m$. The result then follows from the fact that, for each $(k_1, k_2) \in K \times K$ and each g_2 in a neighborhood of k_2 in $G|K$, $g_1 \to \omega(g_1 g_2)m$ and $g_1 \to \omega(g_1)\omega(g_2)m$ are both exponential functions in a neighborhood of k_1 in $G|K$ with value $\omega(k_1)\omega(g_2)m$ at k_1.

7.2. Proposition. *Let (M, ω) be an analytic DNF $(\mathfrak{g}, K)$-module. Then the map $\theta\colon \mathcal{O}_{G|K} \hat{\otimes} M \to \mathcal{O}_{G|K} \hat{\otimes} M$, defined above, is an $\mathcal{O}_{G|K}$-module automorphism with the following additional properties:*

(1) $\theta[(\xi_l \hat{\otimes} 1)f] = [\xi_l \hat{\otimes} 1 + 1 \hat{\otimes} \omega(\xi)]\, \theta f$,

(2) $\theta[(\xi_r \hat{\otimes} 1 + 1 \hat{\otimes} \omega(\xi))f] = (\xi_r \hat{\otimes} 1)\, \theta f$, *and*

(3) $\theta[(r(k) \hat{\otimes} \omega(k))f] = (r(k) \hat{\otimes} 1)\, \theta f$

for $f \in \mathcal{O}_{G|K} \hat{\otimes} M$, $k \in K$, *and* $\xi \in \mathfrak{g}$, *where* r *is the right regular representation of* K *on* $\mathcal{O}_{G|K}$. *Furthermore, any* $(\mathfrak{g}, K)$*-module for which there exists a map* θ *with these properties is an analytic* $(\mathfrak{g}, K)$*-module.*

Proof. By construction, θ is an $\mathcal{O}_{G|K}$-module endomorphism. The condition that $\theta(1 \hat{\otimes} m)$ be exponential gives that, for $\xi \in \mathfrak{g}$,

$$\begin{aligned}
&[(\xi_l \hat{\otimes} 1) + (1 \hat{\otimes} \omega(\xi))]\, \theta(h \hat{\otimes} m) \\
&\quad = \xi_l h \cdot \theta(1 \hat{\otimes} m) + h \cdot [(\xi_l \hat{\otimes} 1) + (1 \hat{\otimes} \omega(\xi))] \theta(1 \hat{\otimes} m) \\
&\quad = \xi_l h \cdot \theta(1 \hat{\otimes} m) = \theta[(\xi_l \hat{\otimes} 1)(h \hat{\otimes} m)].
\end{aligned}$$

Thus, (1) is established.

The map θ is an isomorphism and, in fact, its inverse is given by $\tau\theta\tau$, where τ is the unipotent map $h \hat{\otimes} m \to h' \hat{\otimes} m$ with $h'(g) = h(g^{-1})$. To see this, note that

$$\theta \circ \tau\theta\tau(1 \hat{\otimes} m)(g) = \theta \circ \tau\theta(1 \hat{\otimes} m)(g) = \omega(g)\omega(g^{-1})m = 1 \hat{\otimes} m;$$

since θ and $\tau\theta\tau$ are both $\mathcal{O}_{G|K}$-module endomorphisms, we conclude that $\theta\tau\theta\tau = \mathrm{id}$.

Note that $\tau(\xi_l \hat{\otimes} 1)\tau = \xi_r \hat{\otimes} 1$. To establish property (2) we use this fact, property (1), and the fact that $\tau\theta\tau = \theta^{-1}$:

$$\begin{aligned}
&\theta[(\xi_r \hat{\otimes} 1 + 1 \hat{\otimes} \omega(\xi)) f] \\
&\quad = \theta\tau[(\xi_l \hat{\otimes} 1 + 1 \hat{\otimes} \omega(\xi))\, \xi\tau f] \\
&\quad = \theta\tau[(\xi_l \hat{\otimes} 1 + 1 \hat{\otimes} \omega(\xi))\, \theta\tau\theta f] = \theta\tau\theta[(\xi_l \hat{\otimes} 1)\, \tau\theta f] \\
&\quad = \tau(\xi_l \hat{\otimes} 1)\, \tau\theta f = (\xi_r \hat{\otimes} 1)\, \theta f.
\end{aligned}$$

This establishes (2).

To prove (3), note that $\theta[(r(k) \hat{\otimes} \omega(k))(f \hat{\otimes} m)(g)] = f(gk)\omega(g)\,\omega(k)m = f(gk)\omega(gk)m = (r(k) \hat{\otimes} 1)\theta(f \hat{\otimes} m)(g)$.

The final statement of the proposition is evident, since, given such a θ and any $m \in M$, $\theta(1 \hat{\otimes} m)$ will be an exponential function in $\mathcal{O}_{G|K} \hat{\otimes} M$ which agrees with $k \to \omega(k)m$ on K.

If a $(\mathfrak{g}, K)$-module M also has specified a central $U(\mathfrak{h})$-action consistent with the Harish-Chandra homomorphism (cf. Sect. 1), then it has the structure of a $U_{\mathfrak{h}}$-module. If, in addition, the $U(\mathfrak{h})$-action is by the infinitesimal character $\lambda \in \mathfrak{h}^*$, then M is a U_λ-module. In this case, we will call M a (U_λ, K)*-module.*

We will be interested in analytic $(\mathfrak{g}, K)$-modules primarily in the context of representations of semisimple Lie groups. Unless stated to the contrary,

G_0 will denote a connected real semisimple Lie group with finite center. We use the standard notational convention: The Lie algebra of a group is denoted by the corresponding lower case german letter. Symbols for real groups or algebras carry a subscript "0" which is dropped when we pass to complexification. We fix a maximal compact subgroup K_0 in G_0. This choice is essentially unique, as all of them are conjugate. Let ω be a (continuous) representation of G_0 on a complete locally convex topological space V_ω. The space V of K_0-finite vectors is always dense in V_ω. We will assume that ω is *admissible*, i.e., each K-isotypic component occurs in V with finite multiplicity, and of finite length. Then V consists entirely of analytic vectors, is $\mathfrak{g}_0$-invariant, and hence $\mathfrak{g}$-invariant [17]. This is a typical example of a *Harish-Chandra module* (for G_0), which, by definition, is a $(\mathfrak{g}, K_0)$-module of finite length. We refer to V_ω as a *globalization* of V. Every Harish-Chandra module admits a variety of globalizations. In particular, it always admits a globalization on a Hilbert space [10, 28].

The compact group K_0 is contained in its complexification K [6, Chap. III, Sect. 6, Def. 4], and every Harish-Chandra module V extends to a $(\mathfrak{g}, K)$-module. The action of K on V is algebraic, which explains the use of algebraic methods in the theory. We note, that except for the finite dimensional case, V is never an analytic module, and cannot be lifted to a representation of G_0.

There exist certain canonical globalizations of V: the C^∞ globalization introduced by Casselman and Wallach (cf. [40]) and globalizations $V_{\max}$ and $V_{\min}$ defined by Schmid [33]. We will be mainly concerned in this paper with $V_{\min}$. In order to state the main result, we recall that the space of analytic vectors in any Banach globalization V_ω of V is dense, is G_0-invariant, and, with the topology of a subspace of $\mathcal{O}(G_0) \hat{\otimes} V_\omega$, becomes a globalization in its own right.

7.3. Proposition (Schmid [33]). *Let V be a Harish-Chandra module for G_0. Then*

(a) *each Harish-Chandra module V has a canonical globalization $V \to V_{\min}$ with the property that any globalization $V \to V_\omega$ factors as $V \to V_{\min} \to V_\omega$, with $V_{\min} \to V_\omega$ a continuous injection*;

(b) if V_ω *is a Banach space, then $V_{\min} \to V_\omega$ induces a topological isomorphism of $V_{\min}$ onto the space of analytic vectors in V_ω*;

(c) *the correspondence $V \to V_{\min}$ is an exact functor.*

The module $V_{\min}$ is called the *minimal globalization* of V. Though not stated explicitly in [33], it is immediate from the construction that $V_{\min}$ is a DNF space. Thus $V_{\min}$ is an analytic DNF $(\mathfrak{g}, G_0)$-module.

8. Analytic Sheaves of Modules

In this section we investigate the special properties of sheaves which arise from localizing the analytic DNF (U_λ, K)-modules introduced in Section 7. As we shall see below, if M is such a module, then the map $\theta: \mathcal{O}_{G|K} \hat{\otimes} M \to \mathcal{O}_{G|K} \hat{\otimes} M$ of Proposition 7.2 is the key to understanding what happens when we localize M. Inductive arguments which will be needed later make it necessary for us to work with sheaves of modules on an arbitrary variety of parabolics Y, although our main interest is in the flag variety X. We refer the reader to Section 6 for notation and terminology concerning this setup.

Let Y be a variety of parabolic subalgebras of $\mathfrak{g}$ of a given type, and $\mathcal{O}_Y$ the sheaf of germs of holomorphic functions on Y. Let $\mathcal{M}$ be a DNF sheaf on Y which is simultaneously a sheaf of modules over $\mathcal{O}_Y$ and over the Lie algebra $\mathfrak{g}$, such that $\xi(f\mu) = f(\xi\mu) + \xi(f)\mu$ for $\xi \in \mathfrak{g}$, $f \in \mathcal{O}_Y$, and $\mu \in \mathcal{M}$. In other words, we assume that $\mathcal{M}$ is a module over $\mathcal{G}_Y$, or, equivalently, over $U(\mathcal{G}_Y)$ (cf. Sect. 6). In what follows we identify an $\mathcal{A}_\lambda(Y)$-module with the underlying $U(\mathcal{G}_Y)$-module.

Consider the sheaf $\mathcal{D}_G \hat{\boxtimes} U(\mathcal{G}_Y)$, which by definition is the exterior tensor product $(\pi_1^{-1}\mathcal{D}_G) \hat{\otimes} (\pi_2^{-1} U(\mathcal{G}_Y))$, where $\pi_1: G \times Y \to G$ and $\pi_2: G \times Y \to Y$ are the projections. We define the shear transformation $s: G \times Y \to G \times Y$ by $s(g, y) = (g, gy)$. We shall construct a map $\Theta = \Theta_Y: \mathcal{D}_G \hat{\boxtimes} U(\mathcal{G}_Y) \to s^{-1}(\mathcal{D}_G \hat{\boxtimes} U(\mathcal{G}_Y))$ that is strongly related to the map θ of Proposition 7.2.

A choice of a base point y^0 in Y identifies Y with G/P_{y^0}. Then $U(\mathcal{G}_Y)$ may be represented as the direct image under the projection $Y \to G/P_{y^0}$ of the sheaf of holomorphic differential operators on G which are right P_{y^0}-invariant. Similarly, $\mathcal{D}_G \hat{\boxtimes} U(\mathcal{G}_Y)$ may be represented as the direct image under $G \times G \to G \times G/P_{y^0}$ of the sheaf of holomorphic differential operators on $G \times G$ that are invariant under the right action of P_{y^0} on the second factor of $G \times G/P_{y^0}$. The shear transformation $s: G \times Y \to G \times Y$ induces a morphism of sheaves of algebras $\Theta: \mathcal{D}_G \hat{\boxtimes} U(\mathcal{G}_Y) \to s^{-1}(\mathcal{D}_G \hat{\boxtimes} U(\mathcal{G}_Y))$ in the following way: with Y represented as G/P_{y^0}, we define the shear transformation $s_0: G \times G \to G \times G$ by $s_0(g_1, g_2) = (g_1, g_1 g_2)$. This induces a transformation σ_0 of the sheaf of germs of holomorphic functions on $G \times G$ by $\sigma_0(f) = f \circ s_0^{-1}$ and a corresponding transformation Θ_0 of the sheaf of holomorphic differential operators on $G \times G$ by $\Theta_0(\xi) = \sigma_0 \xi \sigma_0^{-1}$. Clearly Θ_0 preserves the sheaf of right P_{y^0}-invariant operators. The morphism $\Theta: \mathcal{D}_G \hat{\boxtimes} U(\mathcal{G}_Y) \to s^{-1}(\mathcal{D}_G \hat{\boxtimes} U(\mathcal{G}_Y))$ is just the direct image of Θ_0 under the projection $G \times G \to G \times G/P_{y^0}$. Note that Θ satisfies

$$\Theta f = f \circ s^{-1}, \qquad \text{for } f \in \mathcal{O}_{G \times Y}, \tag{8.1}$$

$$\Theta(\xi_1 \hat{\boxtimes} 1) = \xi_1 \hat{\boxtimes} 1 + 1 \hat{\boxtimes} \xi, \qquad \text{for } \xi \in \mathfrak{g}, \tag{8.2}$$

$$\Theta(\xi_r \hat{\boxtimes} 1 + 1 \hat{\boxtimes} \xi) = \xi_r \hat{\boxtimes} 1, \qquad \text{for } \xi \in \mathfrak{g}, \tag{8.3}$$

where ξ_r and ξ_l are as in the previous section. Formulas (8.1)–(8.3) completely characterize Θ. In fact the operators of the form $f \hat{\otimes} 1$ and $\xi_l \hat{\otimes} 1$ generate $\mathscr{D}_G \hat{\otimes} 1$, and this subalgebra, along with the operators of the form $\xi_r \hat{\otimes} 1 + 1 \hat{\otimes} \xi$ for $\xi \in \mathfrak{g}$, generate $\mathscr{D}_G \hat{\otimes} U(\mathfrak{g})$. In particular it follows that Θ is independent of the choice of the base point y_0. It is also clear that Θ commutes with direct images: Let $G \times Y \to G \times Y'$ be induced by the canonical projection $Y \to Y'$. Then $\mathscr{D}_G \hat{\boxtimes} U(\mathscr{G}_{Y'})$ is naturally isomorphic to the direct image of $\mathscr{D}_G \hat{\boxtimes} U(\mathscr{G}_Y)$, and $\Theta_{Y'}$ is naturally isomorphic to the direct image of Θ_Y. Note that in the case of $Y = \{\text{point}\}$, $\mathscr{D}_G \hat{\boxtimes} U(\mathscr{G}_Y) = \mathscr{D}_G \hat{\boxtimes} U(\mathfrak{g})$, which acts naturally on $\mathcal{O}_{G|K} \hat{\otimes} M$.

All the above statements, properly interpreted, make sense relative to K: in particular, Θ defines a morphism $\mathscr{D}_{G|K} \hat{\boxtimes} U(\mathscr{G}_Y) \to s^{-1}(\mathscr{D}_{G|K} \hat{\boxtimes} U(\mathscr{G}_Y))$ compatible with direct images ($\mathscr{D}_{G|K}$ denotes the inverse image of $\mathscr{D}_G$ under $\phi: K \to G$ and s stands for the map $(G|K) \times Y \to (G|K) \times Y$). Note that in the case of $Y = \{\text{point}\}$, $\mathscr{D}_{G|K} \hat{\boxtimes} U(\mathscr{G}_Y) = \mathscr{D}_{G|K} \hat{\boxtimes} U(\mathfrak{g})$, which acts naturally on $\mathcal{O}_{G|K} \hat{\otimes} M$ for any analytic DNF $(\mathfrak{g}, K)$-module M.

It will be convenient from a notational point of view to think of Θ as also acting on a class of operators arising from the K-action. That is, if r is the right representation of K on $\mathcal{O}_{G|K}$ and ω is the representation of K on M, then $r \hat{\otimes} \omega$ and $r \hat{\otimes} 1$ are representations of K on $\mathcal{O}_{G|K} \hat{\otimes} M$. In view of Proposition 7.2, it is natural to set

$$\Theta(r \hat{\otimes} \omega) = r \hat{\otimes} 1. \tag{8.4}$$

The following is a restatement of Proposition 7.2:

8.1. Proposition. *Let M be an analytic* $(\mathfrak{g}, K)$*-module. Then the map $\theta: \mathcal{O}_{G|K} \hat{\otimes} M \to \mathcal{O}_{G|K} \hat{\otimes} M$ of Proposition 7.2 is covariant with Θ in the sense that $\theta(\xi\mu) = \Theta(\xi)\theta(\mu)$ and $\theta((r \hat{\otimes} \omega)(k)\mu) = \Theta((r \hat{\otimes} \omega)(k))\theta(\mu) = (r(k) \hat{\otimes} 1)\theta(\mu)$ for $\xi \in \mathscr{D}_G \hat{\otimes} U_\lambda$, $k \in K$, and $\mu \in \mathcal{O}_{G|K} \hat{\otimes} M$.*

When we localize an analytic $(\mathfrak{g}, K)$-module M we obtain a sheaf $\mathscr{M}$ which inherits from M an infinitesimal analytic G-action and a compatible K-action. To state this precisely requires using the map θ. As we shall see below, this map induces an analogous sheaf map $\theta_X: \mathcal{O}_{G|K} \hat{\boxtimes} \mathscr{M} \to s^{-1}(\mathcal{O}_{G|K} \hat{\boxtimes} \mathscr{M})$ which embodies the infinitesimal analytic G-action and the K-action. This leads to the notion of analytic $(\mathscr{G}_X, K)$-module and, more generally, of analytic $(\mathscr{G}_Y, K)$-module for any variety of parabolics Y.

As before, let $\pi_1: G \times Y \to G$ and $\pi_2: G \times Y \to Y$ denote the projections. If $\mathscr{M}$ is a sheaf of $\mathscr{G}_Y$-modules on Y, then $\pi_2^* \mathscr{M} = \mathcal{O}_{G|K} \hat{\boxtimes} \mathscr{M}$ is a sheaf of $\mathscr{D}_{G|K} \hat{\boxtimes} U(\mathscr{G}_Y)$-modules. The shear transformation $(g, y) \to (g, gy)$ for $G \times Y$ (or $G|K \times Y$) is denoted s_Y.

8.2. Definition. Let $\mathcal{M}$ be a DNF sheaf of $U(\mathcal{G}_Y)$-modules. We shall say that $\mathcal{M}$ is an *analytic* $(\mathcal{G}_Y, K)$*-module* if there is an isomorphism of sheaves $\theta_Y: \mathcal{O}_{G|K} \hat{\boxtimes} \mathcal{M} \to s_Y^{-1}(\mathcal{O}_{G|K} \hat{\boxtimes} \mathcal{M})$ which is covariant with Θ_Y in the sense that it satisfies

(a) $\theta_Y(\xi\mu) = \Theta_Y(\xi)\theta(\mu)$ for $\xi \in \mathcal{D}_{G,K} \hat{\otimes} (U(\mathcal{G}_Y))_y$, $\mu \in (\mathcal{O}_{G|K})_k \hat{\otimes} \mathcal{M}_y$,

and

(b) $\theta_Y((r \hat{\otimes} \omega)(k)\mu) = \Theta_Y((r \hat{\otimes} \omega)(k))\theta(\mu) = (r(k) \hat{\otimes} 1)\theta_Y(\mu)$ for $k \in K$ and $\mu \in (\mathcal{O}_{G|K})_k \hat{\otimes} \mathcal{M}_y$,

where $\omega(k): \mathcal{M} \to k^{-1}\mathcal{M}$ is defined from θ_Y by setting $\omega(k)m = \theta_Y(1 \hat{\otimes} m)(k)$ for $m \in \mathcal{M}_y$ and $k \in K$.

If, in addition, $\mathcal{M}$ is an $\mathcal{A}_\lambda(Y)$-module, then we shall call it an *analytic* $(\mathcal{A}_\lambda(Y), K)$*-module.*

Note that, in this definition, the existence of θ implies the existence of the action ω of K on the sheaf $\mathcal{M}$, while condition (b) implies that this action is associative. In fact, one can define a notion of a sheaf with K-action in terms of a morphism $\mathcal{O}_K \hat{\boxtimes} \mathcal{M} \to s_Y^{-1}(\mathcal{O}_K \hat{\boxtimes} \mathcal{M})$ which satisfies (b) (this is equivalent to the definition in [27, p. 30]). Our definition contains more information than this, however. The analyticity is reflected in "thickening" $\mathcal{O}_K$ to $\mathcal{O}_{G|K}$. Moreover, in condition (a), we require compatibility of the "thickened" K-action with the infinitesimal action of $U(\mathcal{G}_Y)$.

We return now to the case of $Y = X$.

8.3. Proposition. *Let M be a DNF analytic (U_λ, K)-module with separated pth $\mathfrak{n}_x$-homology for each x and let $\mathcal{M} = \Delta_p(M)$. Then $\mathcal{M}$ is an analytic $(\mathcal{D}_\lambda, K)$-module.*

Proof. We may compute $\Delta_p(M)$ using the Koszul complex $K(\mathcal{N}, M)$ as in Proposition 2.4. The fact that M is analytic allows us to use essentially the same idea as in Lemma 4.3 to locally trivialize this complex of bundles. Given $x_0 \in X$, let γ be a local cross section, in a neighborhood of x_0, for $\pi: G \to X$, where $\pi(g) = gx_0$, and let $\theta^\wedge: \mathcal{O}_{x_0} \hat{\otimes} M \to \mathcal{O}_{x_0} \hat{\otimes} M$ be the $\mathcal{O}_{x_0}$-module isomorphism given by $\theta^\wedge f = (\theta(f \circ \pi)) \circ \gamma$. The morphism $\Theta: \mathcal{D}_G \hat{\otimes} U_\lambda \to \mathcal{D}_G \hat{\otimes} U_\lambda$ maps $\mathcal{O}_G \hat{\otimes} U_\lambda$ to itself and we can use the same trick to define an $\mathcal{O}_{x_0}$-module isomorphism $\Theta^\wedge: \mathcal{O}_{x_0} \hat{\otimes} U_\lambda \to \mathcal{O}_{x_0} \hat{\otimes} U_\lambda$; i.e., we set $\Theta^\wedge f = (\Theta(f \circ \pi)) \circ \gamma$. Then, from the definition of Θ, we see that $\Theta^\wedge(\mathcal{O}_{x_0} \hat{\otimes} \mathfrak{n}_{x_0}) = \mathcal{N}_{x_0}$. In this manner, we obtain an isomorphism of complexes $\mathcal{O}_{x_0} \hat{\otimes} K(\mathfrak{n}_{x_0}, M) \to K(\mathcal{N}, M)$, which in degree p is $(\bigwedge^p_{\mathcal{O}_{x_0}} \Theta) \hat{\otimes}_{\mathcal{O}_{x_0}} \theta$. It follows that $K(\mathcal{N}, M)$ has Hausdorff pth homology at x_0 because M has Hausdorff pth $\mathfrak{n}_x$-homology at every x. Since $\Delta_p(M)$ is a direct summand of the pth homology of $K(\mathcal{N}, M)$, it follows from Proposition 3.3 that $\Delta_p(M)$ is a DNF sheaf of modules.

We define the isomorphism $\theta\colon \mathcal{O}_{G|K} \hat{\boxtimes} \mathcal{M} \to s^{-1}(\mathcal{O}_{G|K} \hat{\boxtimes} \mathcal{M})$ as follows: By restriction, the map $\mathcal{D}_{G|K} \hat{\boxtimes} U(\mathcal{G}_X) \to s^{-1}(\mathcal{D}_{G|K} \hat{\boxtimes} U(\mathcal{G}_X))$ induces a map $\mathcal{O}_{G|K} \hat{\boxtimes} \mathcal{N} \to s^{-1}(\mathcal{O}_{G|K} \hat{\boxtimes} \mathcal{N})$ covariant with $f \to f \circ s^{-1}\colon O_{G\times X} \to s^{-1}O_{G\times X}$. Thus, for each p, Θ_X induces an isomorphism from $\bigwedge^0_{\mathcal{O}_{G|K} \hat{\otimes} \mathcal{O}_X}(\mathcal{O}_{G|K} \hat{\boxtimes} \mathcal{N}) = \mathcal{O}_{G|K} \hat{\boxtimes} (\bigwedge^p_{\mathcal{O}_X} \mathcal{N})$ to $\bigwedge^p_{s^{-1}(\mathcal{O}_{G|K} \hat{\otimes} \mathcal{O}_X)} s^{-1}(\mathcal{O}_{G|K} \hat{\boxtimes} \mathcal{N}) = s^{-1}(\mathcal{O}_{G|K} \hat{\boxtimes} (\bigwedge^p_{\mathcal{O}_X} \mathcal{N}))$ which, on tensoring relative to $\mathcal{O}_G$ with $\theta\colon \mathcal{O}_{G|K} \hat{\otimes} M \to \mathcal{O}_{G|K} \hat{\otimes} M$, yields an isomorphism $\mathcal{O}_{G|K} \hat{\boxtimes} (\bigwedge^p_{\mathcal{O}_X} \mathcal{N}) \hat{\otimes} M \to s^{-1}(\mathcal{O}_{G|K} \hat{\boxtimes} (\bigwedge^p_{\mathcal{O}_X} \mathcal{N})) \hat{\otimes} M$. This determines an isomorphism of complexes $\mathcal{O}_{G|K} \hat{\boxtimes} K(\mathcal{N}, M) \to s^{-1}(\mathcal{O}_{G|K} \hat{\boxtimes} K(\mathcal{N}, M))$ and, since the pth homology of $K(\mathcal{N}, M)$ is a DNF sheaf, an isomorphism $\mathcal{O}_{G|K} \hat{\boxtimes} \Delta_p(M) \to s^{-1}(\mathcal{O}_{G|K} \hat{\boxtimes} \Delta_p(M))$. This is covariant with Θ, in the sense of Definition 8.2, by construction.

Now let Y be any variety of parabolics and let $\pi\colon X \to Y$ be the canonical projection.

8.4. PROPOSITION. *If $\mathcal{M}$ is an analytic $(\mathcal{G}_X, K)$-module, then $\pi_*(\mathcal{M})$ is an analytic $(\mathcal{G}_Y, K)$-module, as is $D^p\pi_*(\mathcal{M})$ for $p>0$ provided each of its stalks is Hausdorff. In particular, if $\mathcal{M}$ is an analytic $(\mathcal{D}_\lambda, K)$-module, then $\pi_*(\mathcal{M})$ is an analytic $(\mathcal{A}_\lambda(Y), K)$-module, as is $D^p\pi_*(\mathcal{M})$ for $p>0$ if each of its stalks is Hausdorff.*

Proof. It is clear from the definition that if $\mathcal{M}$ is a DNF sheaf then so is $\pi_*(\mathcal{M})$.

Let U be a compact neighborhood in K, let $\mathcal{O}_{G|U}$ be the restriction of $\mathcal{O}_{G|K}$ to U, and let $\mathscr{C}$ be the Čech resolution functor for sheaves on $U\times X$ based on a projection $Z \to U\times X$, where Z is compact and totally disconnected (cf. Sect. 3). For each p we may apply $\mathscr{C}^p$ to $\mathcal{O}_{G|U} \hat{\boxtimes} \mathcal{M}$ as well as to $s_X^{-1}(\mathcal{O}_{G|U} \hat{\boxtimes} \mathcal{M})$ and in each case the resulting sheaf is naturally a module over $\mathcal{D}_{G|U} \hat{\boxtimes} U(\mathcal{G}_X)$. Evidently, the morphism

$$\theta\colon \mathscr{C}^p(\mathcal{O}_{G|U} \hat{\boxtimes} \mathcal{M}) \to \mathscr{C}^p(s_X^{-1}(\mathcal{O}_{G|U} \hat{\boxtimes} \mathcal{M}))$$

is Θ_X-covariant, which implies that, upon passing to the direct image under $\pi_U\colon U\times X \to Y$, we obtain a Θ_Y-covariant morphism

$$D^p(\pi_U)_*(\mathcal{O}_{G|U} \hat{\boxtimes} \mathcal{M}) \to D^p(\pi_U)_*(s_X^{-1}(\mathcal{O}_{G|U} \hat{\boxtimes} \mathcal{M}))$$

$$\cong s_Y^{-1}(D^p(\pi_U)_*(\mathcal{O}_{G|U} \hat{\boxtimes} \mathcal{M})).$$

For appropriately chosen small compact neighborhoods U, the sheaf $\mathcal{O}_{G|U}$ is acyclic for Γ. Assuming that $D^p\pi_*(\mathcal{M})$ has Hausdorff stalks and using the Künneth formula, we obtain a morphism $\Gamma(\mathcal{O}_{G|U}) \hat{\otimes} D^p\pi_*(\mathcal{M}) \to$

$s_Y^{-1}(\Gamma(\mathcal{O}_{G|U}) \hat{\otimes} D^p\pi_*(\mathcal{M}))$ for each such set U. On passing to germs, we have a morphism

$$\theta_Y: \mathcal{O}_{G|K} \hat{\boxtimes} D^p\pi_*(\mathcal{M}) \to s_Y^{-1}(\mathcal{O}_{G|K} \hat{\boxtimes} D^p\pi_*(\mathcal{M}))$$

which clearly satisfies the conditions of Definition 8.2.

In the case where $Y = \{\text{pt.}\}$, this proposition says that the module of global sections and each cohomology module (if Hausdorff) of an analytic $(\mathcal{G}_X, K)$-module is an analytic $(\mathfrak{g}, K)$-module. That the $\mathfrak{g}$-module structure determined by θ as in Proposition 7.2 is the same as that which is induced by the $U(\mathcal{G}_X)$-module structure of $\mathcal{M}$ follows from the fact that θ is covariant with Θ.

We would also like to know that the partial localization functors of Section 6 preserve analytic modules. The proof that this is so is almost identical to the proof of Proposition 8.3.

8.5. Proposition. *If $\mathcal{H}$ is an analytic $(\mathcal{A}_\lambda(Y), K)$-module, then $D_p \Delta_\lambda^Y(\mathcal{H})$ is an analytic $(\mathcal{D}_\lambda, K)$-module for each p provided it has Hausdorff geometric fiber at each point.*

Proof. Recall from Section 6 that $K(\mathcal{N}_u, \mathcal{H})$ denotes the Koszul complex for $\mathcal{N}_u = \mathcal{N}/\pi^*(\mathcal{U}_Y)$ acting on $\mathcal{H}$, and $H_k(\mathcal{N}_u, \mathcal{H}) = H_k(K(\mathcal{N}_u, \mathcal{H}))$ denotes its kth homology. Here, $\mathcal{U}_Y$ is the sheaf of local sections of the bundle of nilpotent radicals of the bundle of parabolics corresponding to points of Y. By Proposition 6.3, $D_k \Delta_\lambda^Y(\mathcal{H})$ is naturally isomorphic to the $(\lambda+\rho)$-weight subspace of $H_k(\mathcal{N}_u, \mathcal{H})$. As in the proof of Proposition 8.3, we may use θ_Y to locally trivialize $K(\mathcal{N}_u, \mathcal{H})$ and, hence, conclude that $D_k \Delta_\lambda^Y(\mathcal{H})$ has Hausdorff stalks and is therefore DNF provided it has Hausdorff geometric fiber at each point. In what follows below, we assume that this is the case.

The isomorphism $\Theta: \mathcal{D}_{G|K} \hat{\boxtimes} U(\mathcal{G}_Y) \to s_Y^{-1}(\mathcal{D}_{G|K} \hat{\boxtimes} U(\mathcal{G}_Y))$ maps $\mathcal{O}_{G|K} \hat{\boxtimes} \pi^*(\mathcal{U}_Y)$ to $s_Y^{-1}(\mathcal{O}_{G|K} \hat{\boxtimes} \pi^*(\mathcal{U}_Y))$ and, hence, determines an isomorphism $\mathcal{O}_{G|K} \hat{\boxtimes} \mathcal{N}_u \to s^{-1}(\mathcal{O}_{G|K} \hat{\boxtimes} \mathcal{N}_u)$ which is covariant with $f \to f \circ s^{-1}$: $(\mathcal{O}_{G|K} \hat{\boxtimes} \mathcal{O}_X \to s^{-1}(\mathcal{O}_{G|K} \hat{\boxtimes} \mathcal{O}_X)$. Similarly, we have an isomorphism $\mathcal{O}_{G|K} \hat{\boxtimes} \bigwedge^p_{\mathcal{O}_X} \mathcal{N}_u \to s^{-1}(\mathcal{O}_{G|K} \hat{\boxtimes} \bigwedge^p_{\mathcal{O}_X} \mathcal{N}_u)$ for each p. On tensoring relative to $\mathcal{O}_{G|K} \hat{\boxtimes} \mathcal{O}_X$ with $\theta_Y: \mathcal{O}_{G|K} \hat{\boxtimes} \mathcal{H} \to \mathcal{O}_{G|K} \hat{\boxtimes} \mathcal{H}$, this yields an isomorphism which is the first column of the commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_{G|K} \hat{\boxtimes} K(\mathcal{N}_u, \mathcal{H}) \arrow[r] \arrow[d] & \mathcal{O}_{G|K} \hat{\boxtimes} \pi^*(\mathcal{H}) \arrow[r] \arrow[d] & 0 \\
s^{-1}(\mathcal{O}_{G|K} \hat{\boxtimes} K(\mathcal{N}_u, \mathcal{H})) \arrow[r] & s^{-1}(\mathcal{O}_{G|K} \hat{\boxtimes} \pi^*(\mathcal{H})) \arrow[r] & 0.
\end{tikzcd}$$

The second column is just θ_Y pulled back to $(G|K)\times X$. The rows of this diagram are Koszul resolutions. On passing to homology, we obtain for each k a map

$$\theta\colon \mathcal{O}_{G|K} \mathbin{\hat{\boxtimes}} H_k(\mathcal{N}_{\mathrm{u}}, \mathcal{H}) \to s^{-1}(\mathcal{O}_{G|K} \mathbin{\hat{\boxtimes}} H_k(\mathcal{N}_{\mathrm{u}}, \mathcal{H}))$$

which is covariant with Θ.

Next, we show how to construct analytic sheaves.

Analytic Induction. Fix $y \in Y$, let Q be the K-orbit of y, and let $\psi\colon G|K \to Y|Q$ be the projection $\psi(g) = gy$. Denote by J_y the isotropy group of y in K and let (σ, L) be an analytic DNF $(\mathfrak{p}_y, J_y)$-module. By $\mathcal{I}'(y, L)$ we shall mean the sheaf of germs $f \in \psi_*(\mathcal{O}_{G|K} \mathbin{\hat{\otimes}} L)$ which satisfy the relation $(1 \mathbin{\hat{\otimes}} \sigma(\xi)) f = -(\xi_{\mathrm{r}} \mathbin{\hat{\otimes}} 1) f$ for $\xi \in U(\mathfrak{p}_y)$, and $(1 \mathbin{\hat{\otimes}} \sigma(k)) f = r(k^{-1}) f$ for k in J_y. Here $r(k)$ denotes the "right translation by k" operator on $\mathcal{O}_{G|K}$. If C is a compact subset of Q, then we topologize $\Gamma(C, \mathcal{I}'(y, L))$ by giving it the topology it inherits from being identified with a closed subspace of the DNF space $\Gamma(C', \mathcal{O}_{G|K} \mathbin{\hat{\otimes}} L)$, where C' is any compact subset of K with $\psi(C') = C$ (the topology is independent of the choice of C'). With this topology, $\Gamma(C, \mathcal{I}'(y, L))$ is a DNF space and $\mathcal{I}'(y, L)$ is a DNF sheaf (cf. Sects. 2 and 3). We denote by $\mathcal{I}(y, L)$ the sheaf on Y obtained from $\mathcal{I}'(y, L)$ by extension by zero. This is again a DNF sheaf. Moreover, it is a $U(\mathcal{G}_Y)$-module: The action is induced from the action of $\mathcal{D}_{G|K}$ on the first factor of $\mathcal{O}_{G|K} \mathbin{\hat{\otimes}} L$. Left translations define an action of K on $\mathcal{I}(y, L)$. Evidently both actions are compatible.

Let $t\colon G \times G \to G \times G$ and $t\colon (G|K) \times (G|K) \to (G|K) \times (G|K)$ be the shear transformation given by $t(g_1, g_2) = (g_1, g_1 g_2)$. The map $\tau\colon \mathcal{O}_G \mathbin{\hat{\boxtimes}} \mathcal{O}_G \to t^{-1}(\mathcal{O}_G \mathbin{\hat{\boxtimes}} \mathcal{O}_G)$, defined by $\tau(f) = f \circ t^{-1}$, induces a map $\mathcal{O}_{G|K} \mathbin{\hat{\boxtimes}} \mathcal{O}_{G|K}$ to $t^{-1}(\mathcal{O}_{G|K} \mathbin{\hat{\boxtimes}} \mathcal{O}_{G|K})$ and, on passing to direct image in the second factor, followed by extension by zero, defines a map $\theta_Y\colon \mathcal{O}_{G|K} \mathbin{\hat{\boxtimes}} \mathcal{I}(y, L) \to s^{-1}(\mathcal{O}_{G|K} \mathbin{\hat{\boxtimes}} \mathcal{I}(y, L))$. This clearly is Θ-covariant. Hence, $\mathcal{I}(y, L)$ is an analytic DNF $(\mathcal{G}_Y, K)$-module supported on Q.

We have not yet used the hypothesis that (σ, L) is an *analytic* module. This comes into play in showing that $\mathcal{I}(y, L)$ is nontrivial. In fact, we shall show that the geometric fiber of $\mathcal{I}(y, L)$ at y is isomorphic to L. Evaluation at e in G gives a map from the geometric fiber of $\mathcal{I}(y, L)_y$ to L and this map is clearly injective. We prove that it is also surjective as follows: Over some neighborhood U of y, we choose a holomorphic section $\gamma\colon U \to G$ for the projection $g \to g_y\colon G \to Y$ with $\gamma(y) = e$. If I is a point of L, then there is an exponential function f_I for I, defined in an infinitesimal neighborhood $P_y|J_y$. The function q defined by $q(g) = f_I(g^{-1}\gamma \circ \pi(g))$ is defined in $G|K$ and satisfies the identities that ensure that it defines a section of $\mathcal{I}(y, L)$ in a neighborhood of y. Since $q(e) = f(e) = I$, the proof of surjectivity is complete.

Let $Y = X$ and suppose that σ vanishes on the nilpotent radical of $\mathfrak{p}_y = \mathfrak{b}_y$. If (σ, L), regarded as a $\mathfrak{h}_y$-module, has infinitesimal character determined by the parameter $\lambda + \rho \in \mathfrak{h}^*$, then $\mathscr{I}(y, L)$ is clearly an analytic $(\mathscr{D}_\lambda, K)$-module.

We summarize the properties of $\mathscr{I}(y, L)$, developed above, in the following proposition:

8.6. PROPOSITION. *For each analytic $(\mathfrak{p}_y, J_y)$-module (σ, L), the sheaf $\mathscr{I}(y, L)$, constructed above, is an analytic $(\mathscr{G}_Y, K)$-module which has support $Q = Ky$, is a locally free $\mathcal{O}_{Y|Q}$-module on Q, and has geometric fiber L at y. If $Y = X$ and σ, as a representation of $\mathfrak{p}_y = \mathfrak{b}_y$, has infinitesimal character $\lambda \in \mathfrak{h}^*$, then $\mathscr{I}(y, L)$ is an analytic $(\mathscr{D}_\lambda, K)$-module.*

We shall call $\mathscr{I}(y, L)$ the $(\mathscr{G}_Y, K)$-module *induced* from L at y.

Now suppose that $\mathscr{M}$ is an analytic $(\mathscr{G}_Y, K)$-module with separated geometric fiber L. Then L is a DNF space and there are natural compatible actions of $\mathfrak{p}_y$ and J_y on L which, as we shall see below, make it an analytic $(\mathfrak{p}_y, J_y)$-module.

8.7. PROPOSITION. *Let Q be a K-orbit in Y and y a point of Q. Let $\mathscr{M}$ be an analytic $(\mathscr{G}_y, K)$-module which has separated geometric fiber L at y. Then L is an analytic $(\mathfrak{p}_y, J_y)$-module and $\mathscr{M}$ is isomorphic on Q to the $(\mathscr{D}_\lambda, K)$-module $\mathscr{I}(y, L)$.*

Proof. Since $\mathscr{M}$ is an analytic $(\mathscr{G}_Y, K)$-module, there is a map $\theta: \mathcal{O}_{G|K} \hat{\boxtimes} \mathscr{M} \to s^{-1}(\mathcal{O}_{G|K} \hat{\boxtimes} \mathscr{M})$ which is covariant with Θ as in Definition 8.2. This induces a map $\theta^y: \mathcal{O}_{P_y|J_y} \hat{\otimes} L \to \mathcal{O}_{P_y|J_y} \hat{\otimes} L$ as follows: s leaves $(P_y | J_y) \times \{y\}$ invariant and so Θ leaves invariant the ideal sheaf of this set in $\mathcal{O}_{G|K} \hat{\boxtimes} \mathcal{O}_Y$. The quotient of $\mathcal{O}_{G|K} \hat{\boxtimes} \mathscr{M}$ modulo the submodule generated by this ideal sheaf is a sheaf supported on $(P_y | J_y) \times \{y\}$ which, via $g \to (g, y)$, pulls back to $\mathcal{O}_{P_y|J_y} \hat{\otimes} L$. Since θ is covariant with Θ, it follows that θ induces a map θ^y as above. Hence, L is analytic.

With the appropriate descriptions of $\mathscr{I}(y, L)$ and $\mathscr{M}$ in terms of $\mathcal{O}_{G|K} \hat{\boxtimes} \mathscr{M}$, it will be apparent that θ induces an isomorphism from $\mathscr{I}(y, L)$ to $\mathscr{M}$.

Recall that $\psi: G|K \to Y|Q$ is the projection $\psi(g) = gy$. Let $\mathscr{I}(y, \mathscr{M})$ be the sheaf on $Y \times Y$ which has support $Q \times Y$ and which, on $Q \times Y$, consists of germs $f \in (\psi \times 1)_*(\mathcal{O}_{G|K} \hat{\boxtimes} \mathscr{M})$ which satisfy the relations

$$(1 \hat{\otimes} \sigma(\xi)) f = -(\xi_r \hat{\otimes} 1) f \qquad \text{for} \quad \xi \in U(\mathfrak{p}_y),$$

and

$$(1 \hat{\otimes} \sigma(k)) f = r(k^{-1}) f \qquad \text{for} \quad k \text{ in } J_y.$$

Here $r(k)$ denotes the "right translation by k" operator on $\mathcal{O}_{G|K}$. Then $\mathcal{I}(y, L) = i_y^* \mathcal{I}(y, \mathcal{M})$, where $i_y: G \to G \times Y$ is the map $g \to (g, y)$.

Now, it follows from the fact that θ is covariant with Θ that θ maps $\mathcal{I}(y, \mathcal{M})$ to the sheaf supported on $Q \times Y$ which, on $Q \times Y$, consists of germs $f \in (\psi \times 1)_* (\mathcal{O}_{G|K} \hat{\boxtimes} \mathcal{M})$ which satisfy the relations

$$(\xi_r \hat{\otimes} 1) f = 0 \qquad \text{for} \quad \xi \in U(\mathfrak{p}_y),$$

and

$$r(k) f = f \qquad \text{for} \quad k \text{ in } J_y.$$

That is, θ maps $\mathcal{I}(y, \mathcal{M})$ to the germs which are constant on the fibers of $\psi \times 1$. This is just a copy of $\mathcal{O}_{Y|Q} \hat{\boxtimes} \mathcal{M}$. Furthermore, since $s \circ i_y$ is the map $\delta_y: G \to G \times Y$ given by $g \to (g, gy)$, we conclude that θ induces an isomorphism from $\mathcal{I}(y, L) = i_y^* \mathcal{I}(y, \mathcal{M})$ to $\delta_y^*(\mathcal{O}_{Y|Q} \hat{\boxtimes} \mathcal{M})$. It remains to show that this latter space is isomorphic to $\mathcal{M}$.

There is a short exact sequence

$$0 \to \mathcal{J}_\Delta \to \mathcal{O}_{Y \times Y} \to \mathcal{O}_\Delta \to 0,$$

where Δ is the diagonal in $Y \times Y$ and $\mathcal{J}_\Delta$ is its ideal sheaf in $\mathcal{O}_{Y \times Y}$. If we tensor this sequence with $\mathcal{O}_Y \hat{\boxtimes} \mathcal{M}$ over $\mathcal{O}_{Y \times Y}$, we conclude that the map $\mathcal{O}_Y \hat{\boxtimes} \mathcal{M} \to \mathcal{M}$, induced by the module operation, has as kernel the sheaf $\mathcal{J}_\Delta(\mathcal{O}_Y \hat{\boxtimes} \mathcal{M})$. That is

$$\mathcal{M} \cong \delta^*(\mathcal{O}_Y \hat{\boxtimes} \mathcal{M}) = \delta^{-1}(\mathcal{O}_Y \hat{\boxtimes} \mathcal{M} / \mathcal{J}_\Delta(\mathcal{O}_Y \hat{\boxtimes} \mathcal{M})),$$

where δ: $Y \to Y \times Y$ is the diagonal map. This completes the proof.

9. Standard Modules and Sheaves

In Section 5 we have shown that the categories $D\mathfrak{M}(U_\lambda)$ and $D\mathfrak{M}(\mathcal{D}_\lambda)$ are equivalent. The equivalence is provided by the derived functors $D\Delta$ and $D\Gamma$. In this section, as in Section 7, we let G_0 be a connected semisimple Lie group with finite center, with complexified Lie algebra $\mathfrak{g}$, and we define the full subcategory $D\mathfrak{M}(U_\lambda, G_0)$ of $D\mathfrak{M}(U_\lambda)$, generated by minimal globalizations of Harish-Chandra modules, as well as the full subcategory $D\mathfrak{M}(\mathcal{D}_\lambda, G_0)$ of $D\mathfrak{M}(\mathcal{D}_\lambda)$, generated by modules which are analytically induced on a G_0-orbit and are of finite rank. One of the main results of this paper is the assertion that the functors $D\Delta$ and $D\Gamma$ establish an equivalence of $D\mathfrak{M}(U_\lambda, G_0)$ and $D\mathfrak{M}(\mathcal{D}_\lambda, G_0)$. We will give a proof of this fact in the next section. Here we define these categories and study their structure. We also investigate the effect of the intertwining functors on $D\mathfrak{M}(\mathcal{D}_\lambda, G_0)$.

We first consider a full subcategory $\mathfrak{M}(U_\lambda, G_0)$ of $\mathfrak{M}(U_\lambda)$ consisting of modules having a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

by modules in $\mathfrak{M}(U_\lambda)$ with consecutive subquotients being minimal globalizations to G_0 of Harish-Chandra (U_λ, K)-modules, where K is the complexification of the maximal compact subgroup of G_0 (cf. Sect. 7).

The fact that it may be possible for two modules which are minimal globalizations of Harish-Chandra (U_λ, K)-modules to have an extension as U_λ-modules which is not a (U_λ, G_0)-module makes it necessary to define our category $\mathfrak{M}(U_\lambda, G_0)$ in this way. Note that, although morphisms in $\mathfrak{M}(U_\lambda, G_0)$ are only required to be U_λ-module homomorphisms, whenever two objects in $\mathfrak{M}(U_\lambda, G_0)$ are actually (U_λ, G_0)-modules, a morphism between them will necessarily also preserve the G_0-action since G_0 is connected.

9.1. LEMMA. *The category $\mathfrak{M}(U_\lambda, G_0)$ is a full abelian subcategory of $\mathfrak{M}(U_\lambda)$ which is closed under extensions.*

Proof. It is immediate from the definition and the facts in Appendix A that $\mathfrak{M}(U_\lambda, G_0)$ is closed under passing to closed U_λ-submodules and separated quotients and is closed under extensions. It is a full subcategory by definition. Thus, to complete the proof, we need only show that a morphism $f: M \to N$ between objects in $\mathfrak{M}(U_\lambda, G_0)$ has an image which belongs to $\mathfrak{M}(U_\lambda, G_0)$. This amounts to showing that f has closed range.

An immediate consequence of the exactness of the minimal globalization functor (Proposition 7.3) is the fact that if M and N are irreducible minimal globalizations then f is either the zero map or an isomorphism.

We now assume only that M is irreducible. We choose a maximal filtration $\{N_k\}$ for N; i.e., the consecutive quotients are (topologically) irreducible modules. There exists a smallest k such that $f(M) \subseteq N_k$. The induced map $f': M \to N_k/N_{k-1}$ is a nonzero morphism of two irreducible modules, and therefore an isomorphism. Define $\varphi: M \oplus N_{k-1} \to N_k$ by $\varphi(v, w) = f(v) + w$. Clearly φ is a continuous algebraic isomorphism between two DNF spaces—hence a topological isomorphism (cf. Appendix A.6). In particular $f(M) = \varphi(M)$ is closed in N_k and, consequently, $f(M)$ is closed in N.

Let us turn now to the general case. We assume that the filtrations of M and N are maximal. We will show that for each k, $f(M_k)$ is closed in N. The statement is obviously true for $k = 0$. Suppose $f(M_{k-1})$ is closed in N and consider the induced map $f': M_k/M_{k-1} \to N/f(M_{k-1})$. Then $N/f(M_{k-1})$ is in $\mathfrak{M}(U_\lambda, G_0)$ and, since M_k/M_{k-1} is an irreducible module,

f' has closed range. Let π be the projection of N onto $N/f(M_{k-1})$. Then $f(M_k)=\pi^{-1}f'(M_k/M_{k-1})$, and is therefore closed in N. This completes the proof of the lemma.

Denote by $D\mathfrak{M}(U_\lambda, G_0)$ the full subcategory of $D\mathfrak{M}(U_\lambda)$, consisting of cohomologically bounded complexes with all cohomology groups in $\mathfrak{M}(U_\lambda, G_0)$.

9.2. PROPOSITION. *The category $D\mathfrak{M}(U_\lambda, G_0)$ is a triangulated subcategory of $D\mathfrak{M}(U_\lambda)$, closed under the functors of upper and lower truncation.*

Proof. Clearly $D\mathfrak{M}(U_\lambda, G_0)$ is invariant under translations. Thus it is enough to show that if $M \to N \to P$ is a distinguished triangle in $D\mathfrak{M}(U_\lambda)$ and M and N are in $D\mathfrak{M}(U_\lambda, G_0)$ then so is P. Now, such a distinguished triangle gives rise to a long exact sequence

$$\cdots \to H^k(M) \to H^k(N) \to H^k(P) \to H^{k+1}(M) \to H^{k+1}(N) \cdots.$$

By assumption, $H^k(M)$ and $H^k(N)$ are in $\mathfrak{M}(U_\lambda, G_0)$ for all k. By Lemma 9.1 the image A^k and kernel B^k of $H^k(M) \to H^k(N)$ belong to $\mathfrak{M}(U_\lambda, G_0)$ for all k. We have an exact sequence $0 \to A^k \to H^k(P) \to B^{k+1} \to 0$. Since $\mathfrak{M}(U_\lambda, G_0)$ is closed under extensions, A.11 implies that $H^k(P)$ belongs to $\mathfrak{M}(U_\lambda, G_0)$. Therefore P is an object in $D\mathfrak{M}(U_\lambda, G_0)$. The statement about truncations is clear.

We now turn our attention to $\mathscr{D}_\lambda$-modules. For $x \in X$ let $B_{0,x}$ be the stabilizer of x in G_0. Let L be a finite dimensional irreducible representation of $B_{0,x}$ such that its differential, regarded as a linear form on $\mathfrak{g}_0 \cap \mathfrak{b}_x$, is obtained from the specialization of $\lambda + \rho$ at x. Then L is automatically an analytic $(\mathfrak{b}_x, B_{0,x})$-module and it determines an induced module $\mathscr{I}(x, L)$ in $\mathfrak{M}(\mathscr{D}_\lambda)$. We will refer to the analytic $(\mathscr{D}_\lambda, G_0)$-module $\mathscr{I}(x, L)$ (cf. Sect. 8) as a *standard* $(\mathscr{D}_\lambda, G_0)$*-module*. We denote by $\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ the full subcategory of $\mathfrak{M}(\mathscr{D}_\lambda)$ consisting of objects $\mathscr{F}$ having a finite filtration

$$0 = \mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \cdots \subseteq \mathscr{F}_m = \mathscr{F}$$

by objects in $\mathfrak{M}(\mathscr{D}_\lambda)$ with associated subquotients isomorphic to standard $(\mathscr{D}_\lambda, G_0)$-modules.

9.3. PROPOSITION. *Let $\mathscr{F}$ be an analytic $(\mathscr{D}_\lambda, G_0)$-module of finite rank. Then $\mathscr{F}$ is an object of $\mathfrak{M}(\mathscr{D}_\lambda, G_0)$.*

Proof. Denote by S the support of $\mathscr{F}$. Let S^0 be the union of all G_0-orbits contained in S of maximal dimension. Let $\mathscr{F}'$ be the sheaf on X obtained by restricting to S^0, and then extending by zero. It follows from

Proposition 8.7 that $\mathscr{F}'$ is a direct sum of $(\mathscr{D}_\lambda, G_0)$-modules of the form $\mathscr{I}(x, L)$ where L is a finite dimensional $G_{0,x}$-module. We have an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$. If $\mathscr{F}'$ and $\mathscr{F}''$ are in $\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ then so is $\mathscr{F}$. We note that support of $\mathscr{F}'$ is of strictly lower dimension than that of S, and we recall that G_0 acts on X with finitely many orbits [41]. A simple induction now implies that $\mathscr{F}$ is an object of $\mathfrak{M}(\mathscr{D}_\lambda, G_0)$, provided this is true of each $(\mathscr{D}_\lambda, G_0)$-module of the form $\mathscr{I}(x, L)$, where L is finite dimensional. The latter follows if we note that a maximal increasing filtration of L by $(\mathfrak{b}_x, B_{0,x})$-modules results in an increasing filtration of $\mathscr{I}(x, L)$ by induced $(\mathscr{D}_\lambda, G_0)$-modules, with standard quotients.

Note that the standard $(\mathscr{D}_\lambda, G_0)$-modules are irreducible objects in the category of all analytic $(\mathscr{D}_\lambda, G_0)$-modules (cf. Sect. 8). They are precisely the irreducible objects in $\mathfrak{M}(\mathscr{D}_\lambda, G_0)$. The category $\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ contains but may be strictly larger than the category of all finite rank analytic $(\mathscr{D}_\lambda, G_0)$-modules, since the latter category may not be closed under extension of $\mathscr{D}_\lambda$-modules.

Let $D\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ be the full subcategory of $D\mathfrak{M}(\mathscr{D}_\lambda)$ of cohomologically bounded complexes with all cohomology groups in $\mathfrak{M}(\mathscr{D}_\lambda, G_0)$.

9.4. Proposition. *The category $\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ is a full abelian subcategory of $\mathfrak{M}(\mathscr{D}_\lambda)$ which is closed under extensions, while the category $D\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ is a triangulated subcategory of $D\mathfrak{M}(\mathscr{D}_\lambda)$, closed under upper and lower truncation functors.*

Proof. We claim that a morphism $\phi\colon \mathscr{M} \to \mathscr{N}$ with $\mathscr{M}$ and $\mathscr{N}$ irreducible objects in $\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ is either the zero morphism or an isomorphism. This statement is trivially true if $\mathscr{M}$ and $\mathscr{N}$ do not have the same support. If they do have the same G_0-orbit S as support, then at each point x of S the stalk map $\phi_x\colon \mathscr{M}_x \to \mathscr{N}_x$ is either zero or an isomorphism. Furthermore, the set of $x \in S$ for which $\phi_x = 0$ is both open and closed in S. Since S is connected, the claim is established. Using this, one can then argue as in the proof of Lemma 9.1 to show that $\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ is an abelian subcategory of $\mathfrak{M}(\mathscr{D}_\lambda)$ which is closed under extensions. The proof that $D\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ is a triangulated subcategory of $D\mathfrak{M}(\mathscr{D}_\lambda)$, closed under upper and lower truncation, follows exactly as in Proposition 9.2.

We want to describe certain canonical sets of generators in the categories $D\mathfrak{M}(U_\lambda, G_0)$ and $D\mathfrak{M}(\mathscr{D}_\lambda, G_0)$. By virtue of the definition, $D\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ has the standard $(\mathscr{D}_\lambda, G_0)$-modules as a set of generators. We want to show that $D\mathfrak{M}(U_\lambda, G_0)$ is generated by minimal globalizations of standard Harish-Chandra modules. There are several constructions of standard Harish-Chandra modules, and properly interpreted, all of them are equivalent [20]. The one that is most suitable for our purpose is that given by

Beilinson and Bernstein (cf. [1, 25]). In order to describe Beilinson–Bernstein standard modules, as well as their minimal globalizations, we first need to dispose of certain important technical details.

We will review some facts on the interplay between G_0 and K orbits on the flag variety X (cf. [24]). Recall that K is the complexification of the maximal compact subgroup K_0 of G_0. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be the Cartan decomposition determined by $\mathfrak{k}$, and τ the corresponding Cartan involution. Let σ be the conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0$. Now, let S be a G_0-orbit. Then there exists $x \in S$, such that $\mathfrak{b}_x$ contains a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$, which is both τ and σ stable. We call such a point x *standard*. All pairs $(x, \mathfrak{c})$ as above, with $x \in S$, are conjugate under K_0. In particular the set of all standard $x \in S$ forms a K_0-orbit. It is equal to the intersection $S \cap Q$ of S and the K-orbit Q containing any standard $x \in S$. In this way one gets a one-to-one correspondence between G_0-orbits and K-orbits.

We want to describe in more detail the isotropy groups $B_{0,x}$, J_x, and $J_{0,x}$ of a standard point x in G_0, K, and K_0, respectively. Let $\mathfrak{c}$ be a σ, τ-stable Cartan subalgebra contained in $\mathfrak{b}_x$. The Cartan subgroup C_0 of G_0 corresponding to $\mathfrak{c}$, i.e., the centralizer of $\mathfrak{c}$ in G_0, is σ, τ-stable, and hence is a direct product of $T_0 = C_0 \cap K_0$ and a vector group A_0. The group T_0 is a maximal reductive subgroup of $J_{0,x}$. We can also choose a maximal reductive subgroup T of J_x so that it has Lie algebra $\mathfrak{t}$, which is the complexification of $\mathfrak{t}_0$. Then $T_0 = T \cap K_0$ and every representation of T_0 extends to T in a unique manner.

9.5. LEMMA. *There is a one-to-one correspondence between*: (1) *finite dimensional representations of* $B_{0,x}$, (2) *finite dimensional* $(\mathfrak{b}_x, C_0)$*-modules*, (3) *finite dimensional* $(\mathfrak{b}_x, J_{0,x})$*-modules*, (4) *finite dimensional* $(\mathfrak{b}_x, T_0)$*-modules*, (5) *finite dimensional holomorphic* $(\mathfrak{b}_x, J_x)$*-modules, and* (6) *finite dimensional holomorphic* $(\mathfrak{b}_x, T)$*-modules*, if *we require in all these cases* $\mathfrak{n}_x$ *to act trivially*.

Proof. A representation of a maximal reductive subgroup can be extended to the group by letting it be trivial on unipotent radical. The last condition implies that representations we refer to are trivial on unipotent radicals, and hence equivalent to representations of maximal reductive subgroups. We need only to add that the group A_0 is connected and simply connected.

Thus we may look at a representation L, as above, in many different contexts. The analytically induced module $\mathscr{I}(x, L)$ is a $(\mathscr{D}_\lambda, G_0)$-module precisely when L satisfies the condition of the lemma, and, in addition, $\mathfrak{c}$ acts on L by specialization of $\lambda + \rho$ to x. In the following we assume this is the case.

We can attach to L an algebraically induced module $\mathscr{I}^{\mathrm{alg}}(x, L)$ over the sheaf of algebras $\mathscr{D}_\lambda^{\mathrm{alg}}$ of algebraic twisted differential operators on X, with support on the closure of the K-orbit Q of x. We briefly sketch this construction. For more details we refer the reader to [1, 2], the appendix in [19], and [25].

We denote by $\mathfrak{M}(\mathscr{D}_\lambda^{\mathrm{alg}}, K)$ the category of coherent $\mathscr{D}_\lambda^{\mathrm{alg}}$-modules equipped with an algebraic K-action, and by $\mathfrak{M}(U_\lambda, K)^{\mathrm{alg}}$ the category of algebraic (U_λ, K)-modules. Then L, regarded as a $(\mathfrak{b}_x, J_x)$-module, canonically determines a K-equivariant connection on Q. Define $\mathscr{I}^{\mathrm{alg}}(x, L)$ in $\mathfrak{M}(\mathscr{D}_\lambda^{\mathrm{alg}}, K)$ to be *the direct image in the category of quasicoherent $\mathscr{D}^{\mathrm{alg}}$-modules* of this connection. We have that $\mathscr{I}^{\mathrm{alg}}(x, L)$ is supported on the closure of Q, is a $(\mathscr{D}_\lambda^{\mathrm{alg}}, K)$-module of finite length, and contains a unique irreducible submodule $\mathscr{J}(x, L)$. Every irreducible object in $\mathfrak{M}(\mathscr{D}_\lambda^{\mathrm{alg}}, K)$ arises in this way. If λ is antidominant (i.e., $2\langle\lambda, \alpha\rangle/\langle\alpha, \alpha\rangle$ is not a nonnegative integer for all $\alpha \in \Delta^+$) and regular, then the functor of global sections on X establishes an equivalence of the category $\mathfrak{M}(\mathscr{D}_\lambda^{\mathrm{alg}}, K)$ with the category $\mathfrak{M}(U_\lambda, K)^{\mathrm{alg}}$. It follows that the (U_λ, K)-module $I(x, L) = \Gamma(\mathscr{I}^{\mathrm{alg}}(x, L))$ contains a unique irreducible submodule, namely $J(x, L) = \Gamma(\mathscr{J}(x, L))$. Conversely, with λ antidominant, every irreducible Harish-Chandra (U_λ, K)-module can be uniquely represented as the unique irreducible submodule of a Harish-Chandra module of this type. If $\mathrm{Re}\langle\lambda, \alpha\rangle \leqslant 0$ for every positive root α (which implies that λ is antidominant) then we shall call a $(\mathfrak{g}, K)$-module of the form $I(x, L)$, as above, a *standard Harish-Chandra* $(\mathfrak{g}, K)$*-module.* Now, let λ be arbitrary but regular, and fix w such that $w\lambda$ satisfies the above negativity condition. A (U_λ, K)-module will be called a *standard Harish-Chandra* (U_λ, K)*-module* if it is a standard Harish-Chandra module when considered as a $(U_{w\lambda}, K)$-module. It turns out that these modules are exactly the standard modules of the Langlands classification (cf. [23, 20]).

9.6. Lemma. *Minimal globalizations of standard Harish-Chandra modules form a set of generators for $D\mathfrak{M}(U_\lambda, G_0)$.*

Proof. Suppose first that λ satisfies: $\mathrm{Re}\langle\lambda, \alpha\rangle \leqslant 0$ for $\alpha \in \Delta^+$. For a (U_λ, K)-module of the form $\Gamma(\mathscr{F})$, $\mathscr{F}$ in $\mathfrak{M}(\mathscr{D}_\lambda^{\mathrm{alg}}, K)$, we define its *length* as the dimension of the support of $\mathscr{F}$, regarded as an algebraic variety. It follows from the discussion above that there is an exact sequence

$$0 \to J(x, L) \to I(x, L) \to H \to 0,$$

with $\mathrm{length}(H) < \mathrm{length}(I(x, L)) = \mathrm{length}(J(x, L))$. The sequence remains exact after applying the functor $(\ )^{\sim}$ of minimal globalization, and we obtain a triangle

$$H^{\sim}[-1] \to J(x, L)^{\sim} \to I(x, L)^{\sim}$$

in $D\mathfrak{M}(U_\lambda, G_0)$. This shows that a given irreducible is in the subcategory generated by globalizations of standard modules together with globalizations of modules of shorter length. Thus, an induction reduces the argument to showing that the globalization of an irreducible M of minimal length is in the subcategory generated by globalizations of standard modules. However, such a module is supported on a closed K-orbit, and therefore it must be a standard Harish-Chandra module.

We can now remove the restriction on λ since, for a general λ, the categories $D\mathfrak{M}(U_\lambda, G_0)$ and $D\mathfrak{M}(U_{w\lambda}, G_0)$ are equivalent, via the isomorphism of U_λ and $U_{w\lambda}$, and then apply the above argument to $D\mathfrak{M}(U_{w\lambda}, G_0)$.

Now, we return to the study of the intertwining functor of Section 6. Let α be a simple root and $\mathscr{T}_{\alpha,\lambda}: D\mathfrak{M}(\mathscr{D}_\lambda) \to D\mathfrak{M}(\mathscr{D}_{s(\alpha)\lambda})$ the intertwining functor corresponding to α. Recall the G-invariant projection $\pi: X \to Y$ of X onto the variety of parabolic subalgebras of type α. For $y \in Y$ let $\mathfrak{p}_y$ be the corresponding parabolic, $\mathfrak{u}_y$ the nilpotent radical of $\mathfrak{p}_y$, and $\mathfrak{l}_y = \mathfrak{p}_y/\mathfrak{u}_y$ the "Levi factor" of $\mathfrak{p}_y$.

9.7. PROPOSITION. *The functor $\mathscr{T}_{\alpha,\lambda}$, restricted to $D\mathfrak{M}(U_\lambda, G_0)$, establishes an equivalence of categories between $D\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ and $D\mathfrak{M}(\mathscr{D}_{s(\alpha)\lambda}, G_0)$.*

Proof. By Corollary 6.5, the functor $\mathscr{T}_{\alpha,\lambda}: D\mathfrak{M}(\mathscr{D}_\lambda) \to D\mathfrak{M}(\mathscr{D}_{s(\alpha)\lambda})$ is an equivalence of categories, with $\mathscr{T}_{\alpha,\lambda}$ being its quasi-inverse. Thus to prove the above proposition it is enough to show that $\mathscr{T}_{\alpha,\lambda}(\mathscr{F})$ is an object of $D\mathfrak{M}(\mathscr{D}_{s(\alpha)\lambda}, G_0)$ if $\mathscr{F}$ is in $D\mathfrak{M}(\mathscr{D}_\lambda, G_0)$. Since $\mathscr{T}_{\alpha,\lambda}$ is a ∂-functor, it is enough to assume that $\mathscr{F}$ is a standard $(\mathscr{D}_\lambda, G_0)$-module. Thus, we assume that $\mathscr{F}$ is of the form $\mathscr{I}(x, L)$.

According to Proposition 6.6 and 9.3 and results in Section 8, it is enough to show that, for each pair (p, q), $D^q\pi_*(\mathscr{F})$ is a DNF sheaf and $D^p\Delta_\alpha(D^q\pi_*(\mathscr{F}))$ is a DNF $(\mathscr{D}_\lambda, G_0)$-module with finite dimensional geometric fibers.

Set $X_y = \pi^{-1}(y)$ for $y = \pi(x)$ and let $i_y: X_y \to X$ be the inclusion map. The geometric fiber of $\mathscr{F}$ along X_y is the sheaf $\mathscr{F}_y = i_y^*\mathscr{F}$. Suppose $\mathscr{F}_y$ has Hausdorff cohomology $H^q(X_y, \mathscr{F}_y)$. Then the stalk of $D^q\pi_*(\mathscr{F})$ at y is Hausdorff—in fact it has $H^q(X_y, \mathscr{F}_y)$ as its geometric fiber. It follows from Propositions 8.4 and 8.7 that $D^q\pi_*(\mathscr{F})$ is an analytic $(\pi_*\mathscr{D}_\lambda, G_0)$-module induced from $H^q(X_y, \mathscr{F}_y)$. Let $\mathfrak{n}_{y,x} = \mathfrak{n}_x/\mathfrak{n}_y$ denote the geometric fiber of the sheaf $\mathscr{N}_\mathfrak{u}$ (cf. Sect. 6). It follows from Proposition 6.3 and Proposition 8.5 that if $\mathscr{H}$ is an analytic $(\pi_*\mathscr{D}_\lambda, G_0)$-module, with Hausdorff geometric fibers, such that its geometric fiber at $y \in Y$ has finite dimensional $\mathfrak{n}_{y,x}$-homology for each y and $x \in X_y$, then $D^p\Delta_\alpha(\mathscr{H})$ is an analytic DNF

$(\mathscr{D}_\lambda, G_0)$-module with finite dimensional geometric fibers. Thus the proof of our proposition reduces to the following statement:

9.8. PROPOSITION. *Let $\mathscr{F} = \mathscr{I}(a, L)$ be a standard $(\mathscr{D}_\lambda, G_0)$-module and set $\mathscr{F}_y = i_y^* \mathscr{F}$, as above. Then for each $y \in Y$ and each $x \in X_y$:*

(a) *$H^q(X_y, \mathscr{F}_y)$ is Hausdorff, and*

(b) *the $\mathfrak{n}_{y,x}$-homology groups of $H^q(X_y, \mathscr{F}_y)$ are finite dimensional.*

Before we prove this, we need the information on the structure of $S \cap X_y$ given by Lemma 9.9 below.

Let $\mathfrak{p}_{y,0} = \mathfrak{p}_y \cap \mathfrak{g}_0$, $\mathfrak{u}_{y,0} = \mathfrak{u}_y \cap \mathfrak{g}_0$. Set $\mathfrak{l}_y = \mathfrak{p}_y/\mathfrak{u}_y$ and $\mathfrak{l}_{y,0} = \mathfrak{p}_{y,0}/\mathfrak{u}_{y,0}$. Denote by $P_{y,0}$ the isotropy group of y in G_0, and by $U_{y,0}$ its unipotent radical. Set $L_{y,0} = P_{y,0}/U_{y,0}$. Then $S \cap X_y$ is just an orbit of the action of $L_{y,0}$ on the flag manifold X_y of $\mathfrak{l}_y$.

9.9. LEMMA. *For each $y \in Y$ and each G_0-orbit S, one of the following holds:*

(a) *The algebra $\mathfrak{l}_{y,0}$ is a real form of $\mathfrak{l}_y$, in which case $S \cap X_y$ is $X_y \cong P^1$, a circle, the complement of a circle, or an open hemisphere.*

(b) *The algebra $\mathfrak{l}_{y,0}$ is contained in the Borel subalgebra of $\mathfrak{l}_y$ corresponding to a point x and contains the corresponding nilradical, in which case $S \cap X_y = \{x\}$.*

(c) *The algebra $\mathfrak{l}_{y,0}$ is contained in the Borel subalgebra of $\mathfrak{l}_y$ corresponding to $x' \neq x$ and contains the corresponding nilradical, in which case $S \cap X_y = X_y \setminus \{x'\}$.*

Proof. Since the conclusions of the lemma are invariant under the G_0-action we may assume that X_y contains a standard x, that is, one for which $\mathfrak{b}_x$ contains a Cartan subalgebra $\mathfrak{c}$ which is both σ and τ invariant. Thus, we assume x is standard and $\mathfrak{c}$ is such a subalgebra. We have $\mathfrak{b}_x = \mathfrak{c} + \mathfrak{n}_x$ where Δ_x^+ is a positive root system for $\mathfrak{c}$ and $\mathfrak{n}_x$ is the sum of the corresponding positive root spaces. Also, $\mathfrak{u}_y$, the unipotent radical of $\mathfrak{p}_y$ is the sum of the positive root spaces not equal to $\mathfrak{g}^{\alpha_x}$ or $\mathfrak{g}^{-\alpha_x}$. We say

(a) α_x is *compact* if $\sigma\alpha_x = -\alpha_x$ (resp. $\tau\alpha_x = \alpha_x$) and $\mathfrak{g}^{\alpha_x} \subset \mathfrak{k}$;

(b) α_x is *noncompact* if $\sigma\alpha_x = -\alpha_x$ (resp. $\tau\alpha_x = \alpha_x$) and $\mathfrak{g}^{\alpha_x} \subset \mathfrak{s}$;

(c) α_x is *real* if $\sigma\alpha_x = \alpha_x$ (resp. $\tau\alpha_x = -\alpha$);

(d) α_x is *complex* if $\sigma\alpha_x \neq \pm\alpha_x$ (resp. $\tau_x\alpha \neq \pm\alpha_x$).

In the first three cases $\mathfrak{l}_y = \mathfrak{c} \oplus \mathfrak{g}^{\alpha_x} \oplus \mathfrak{g}^{-\alpha_x}$ is σ-stable. This implies that σ induces an involution on $\mathfrak{l}_y$ for which $\mathfrak{l}_{y,0}$ is the set of invariants. In this case, $\mathfrak{l}_{y,0}$ is a real form of $\mathfrak{l}_y$. Then the factor of $L_{y,0}$ which acts effectively on X_y is, up to a covering, a real form of $SL_2(\mathbb{C})$, or is equal to $SL_2^{\pm}(R)$. The possible orbits of the action of such a group on the flag manifold $X_y \cong P^1$ are as indicated in (a).

If α_x is complex, then, since either $\sigma(\alpha_x)$ or $-\sigma(\alpha_x)$ is a positive root, exactly one of $\sigma(\mathfrak{g}^{\alpha_x})$ and $\sigma(\mathfrak{g}^{-\alpha_x})$ is a positive root space not equal to $\mathfrak{g}^{\alpha_x}$ or $\mathfrak{g}^{-\alpha_x}$ and, hence, is contained in $\mathfrak{u}_y$. If this is $\sigma(\mathfrak{g}^{\alpha_x})$, then $\{v+\sigma(v): v \in \mathfrak{g}^{\alpha_x}\}$ is contained in $\mathfrak{p}_y \cap \mathfrak{g}_0$ and so its image in $\mathfrak{l}_y$ lies in $\mathfrak{l}_{y,0}$; however, this is also the image of $\mathfrak{g}^{\alpha_x}$ in $\mathfrak{l}_y$. On the other hand, $\mathfrak{l}_{y,0}$ does not contain the image of $\mathfrak{g}^{-\alpha_x}$ in $\mathfrak{l}_y$, since $\mathfrak{g}^{-\alpha_x}$ is not contained in $\mathfrak{u}_y$. Thus, $\mathfrak{l}_{y,0}$ contains the unipotent radical of the Borel subalgebra of $\mathfrak{l}_y$ corresponding to the point x. It is, in turn, contained in this Borel subalgebra, for otherwise it would contain the image of $\mathfrak{g}^{-\alpha_x}$. The $L_{y,0}$-orbit of x in this case is just the point $\{x\}$.

If it is $-\sigma(\alpha_x)$ which is a positive root, then $\mathfrak{l}_{y,0}$ contains the image of $\mathfrak{g}^{-\alpha_x}$ and is contained in the corresponding Borel subalgebra. This is the Borel subalgebra of $\mathfrak{l}_y$ corresponding to another point $x' \in X_y$. In this case, the L_y-orbit of x is the complement of $\{x'\}$.

Proof of Proposition 9.8. The subsets of X_y described in Lemma 9.9 form the possible supports of $\mathscr{F}_y$, which is a locally free finite rank $\mathcal{O}_{X_y}$-module, when restricted to its support. In each case, the cohomology groups $H^q(X_y, \mathscr{F}_y)$ are well understood and are known to be Hausdorff topological vector spaces which vanish except in degrees 0 and 1. This completes the proof of Proposition 9.8(a).

To prove part (b) we will exhibit the $\mathfrak{n}_{y,x}$-homology groups of $H^q(X_y, \mathscr{F}_y)$ as the E_2 term of a certain spectral sequence. Consider the double complex $K^{-p,q} = \Gamma(K_p(\mathfrak{n}_{y,x}, \mathscr{C}^q(\mathscr{F}_y)))$, obtained by applying the Koszul complex for $\mathfrak{n}_{y,x}$ to the Čech resolution of $\mathscr{F}_y$. This double complex is regular, as it lies entirely in the second quadrant. Because cohomology vanishes in degrees other than 0 and 1, the spectral sequence of the first filtration degenerates at stage two. For the two filtrations we have

$$'E_2^{-p,q} = H_p(\mathfrak{n}_{y,x}, H^q(X_y, \mathscr{F}_y))$$

and

$$''E_2^{-p,q} = H^q(X_y, H_p(\mathfrak{n}_{y,x}, \mathscr{F}_y)).$$

Thus, to prove Proposition 9.8(b), it is enough to show that the sheaves $H_p(\mathfrak{n}_{y,x}, \mathscr{F}_y)$ have finite dimensional cohomology on X.

The sheaves $H_p(\mathfrak{n}_{y,x}, \mathscr{F}_y)$ are well understood and are easily computed using standard techniques on the flag manifold of $\mathfrak{sl}_2(\mathbb{C})$ (for completeness of this and several other arguments, we shall carry out these calculations in Appendix B). They are nonvanishing only in degrees 0 and 1, have finite dimensional stalks, and are locally constant on the $L_{y,0}$-orbits of X_y. It follows that each such sheaf has a finite filtration such that each of the corresponding subquotients is supported on a single orbit and is locally constant with finite dimensional stalks on that orbit. Given the possible orbits (Lemma 9.9), such a sheaf obviously has finite dimensional cohomology. This completes the proof of Proposition 9.8.

We end this section by directly computing the action of the intertwining functor $\mathscr{T}_{\alpha,\lambda}$ on a $\mathscr{D}_\lambda$-module $\mathscr{I}(a, L)$ in a situation that will be of crucial importance in the next section.

Let $a \in X$ be a standard point and choose a σ, τ-stable Cartan subalgebra $\mathfrak{c}$ in $\mathfrak{b}_a$. The weight $\lambda + \rho$ in $\mathfrak{b}^*$ specializes to $\lambda_a + \rho_a$ in $\mathfrak{c}^*$. If L is an irreducible $(\mathfrak{c}, C_0)$-module with weight $\lambda_a + \rho_a$, then induction at a gives a $\mathscr{D}_\lambda$-module $\mathscr{I}(a, L)$. On the other hand, let b be the point determined by the condition that $\mathfrak{b}_b$ be the span of $\mathfrak{c}$ and the root spaces for the system of positive roots $(s(\alpha)\,\Delta^+)_a$. Then the weight $s(\alpha)\lambda + \rho$ specializes at b to

$$s(\alpha_b)\,\lambda_b + \rho_b = s(\alpha_b)(\lambda_b + \rho_b - \alpha_b) = \lambda_a + \rho_a - \alpha_a.$$

We obtain a $(\mathfrak{c}, C_0)$-module $L(\alpha)$ with this weight by setting $L(\alpha) = \mathfrak{g}^{\alpha_b} \otimes L$, since $\mathfrak{g}^{\alpha_b}$ has weight $\alpha_b = -\alpha_a$. Thus, we obtain a $\mathscr{D}_{s(\alpha)\lambda}$-module $\mathscr{F}(\alpha) = \mathscr{I}(b, L(\alpha))$ by inducing $L(\alpha)$ from b.

9.10. PROPOSITION. *If α_a is complex, $\sigma\alpha_a \in \Delta_a^+$, and λ is dominant with respect to α (i.e., $2(\lambda, \alpha)/(\alpha, \alpha) \notin -\mathbb{N}$), there is a natural isomorphism $\mathscr{T}_{\alpha,\lambda}(\mathscr{F}) \cong \mathscr{F}(\alpha)$* [1].

Proof. Let S_a and S_b denote the G_0-orbits of a and b, respectively. The choice of α corresponds to case (b) of Lemma 9.9. Thus $S_a \cap X_y = \{a\}$ and $S_b \cap X_y = X_y \setminus \{a\}$. Hence $\mathscr{W} = i_y^* \mathscr{F}$ is supported at $\{a\}$, and $H^p(X, \mathscr{W}) = \mathscr{W}_a$ if $p = 0$, and is zero otherwise. Globally this means that $D^p\pi_*(\mathscr{F}) = 0$ for positive p. In view of Proposition 6.6, our statement is equivalent to the assertion that $D^p\,\Delta_\alpha\pi_*(\mathscr{F})$ vanishes unless $p = -1$, in which case it is isomorphic to $\mathscr{F}(\alpha)$.

Recall that an analytic $(\mathscr{D}_{s(\alpha)\lambda}, G_0)$-module of finite rank is determined on each G_0-orbit by its geometric fiber at a point of the orbit. The sheaf $D^p\,\Delta_\alpha\pi_*(\mathscr{F})$ has geometric fiber at $x \in X_y$ equal to $H_{-p}(\mathfrak{n}_{y,x}, \mathscr{W}_a)^\nu$, where $\nu = (s(\alpha)\lambda)_x + \rho_x$. By G_0-invariance we need only to look at $x = a$ and $x = b$. We claim that

$$H_0(\mathfrak{n}_{y,x}, \mathscr{W}_a)^\vee = 0 \qquad \text{for} \quad x = a, b$$

$$H_1(\mathfrak{n}_{y,x}, \mathscr{W}_a)^\vee = 0 \qquad \text{for} \quad x = a,$$

$$H_1(\mathfrak{n}_{y,x}, \mathscr{W}_a)^\vee = L(\alpha) \text{ as a } (\mathfrak{c}, C_0)\text{-module} \qquad \text{for} \quad x = b.$$

In fact, the assumptions on λ imply that $m = 2\langle \lambda + \rho, \alpha \rangle / \langle \alpha, \alpha \rangle = 2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle + 1$ is not a nonpositive integer. Thus, the statements are consequences of standard $\mathfrak{sl}_2(\mathbb{C})$ $\mathfrak{n}$-homology computations. For completeness, we will do these explicitly in Appendix B.

From the above, it follows that $D^0 \Delta_\alpha \pi_*(\mathscr{F})$ vanishes, while $D^{-1} \Delta_\alpha \pi_*(\mathscr{F})$ is supported on the orbit S_b and is induced from the $(\mathfrak{c}, C_0)$-module $L(\alpha)$ at the point b. Thus, it is naturally isomorphic to $\mathscr{I}(b, L(\alpha))$. This completes the proof.

10. The Main Theorem

In this section we accomplish our main goal, which is to prove that the functors $D\Delta$ and $D\Gamma$ establish an equivalence of the categories $D\mathfrak{M}(U_\lambda, G_0)$ and $D\mathfrak{M}(\mathscr{D}_\lambda, G_0)$. We already know that $D\Delta$ and $D\Gamma$ establish an equivalence of the categories $D\mathfrak{M}(U_\lambda)$ and $D\mathfrak{M}(\mathscr{D}_\lambda)$. So we need only show that $D\Gamma$ maps a generating set of $D\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ to a generating set of $D\mathfrak{M}(U_\lambda, G_0)$. Because of the intertwining functor and Proposition 9.7, it is enough to show this for λ in a given fixed Weyl chamber. We also use the intertwining functor to reduce the problem to looking at induced modules which are supported on orbits of a particular kind and, ultimately, to a construction of discrete series modules due to Schmid [32]. We begin the proof with a discussion of Schmid's result.

We assume that $\operatorname{rank}(G_0) = \operatorname{rank}(K_0)$. By definition, we can then choose a σ, τ-stable Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ contained in $\mathfrak{k}$ (cf. Sect. 9). Any Borel subalgebra $\mathfrak{b}_x$ containing $\mathfrak{c}$ is determined by choice of a set of positive roots $\Delta(\mathfrak{c})^+$ inside the root system $\Delta(\mathfrak{c})$ of $\mathfrak{c}$ in $\mathfrak{g}$. The isotropy group of x in G_0 is the compact Cartan subgroup C_0 of G_0 corresponding to $\mathfrak{c}$. We note that C_0 is contained in K_0. The G_0-orbit S of x is open. The K-orbit Q of x coincides with its K_0-orbit and is therefore closed in X. Every open G_0-orbit (equivalently, every closed K-orbit) arises in this way.

Fix x as above and let $\mathscr{F} = \mathscr{I}(x, L)$ be a standard $(\mathscr{D}_\lambda, G_0)$-module. Since $\lambda_x + \rho_x$ lifts to a character of a compact Cartan subgroup C_0, λ_x is real, in the sense that it has real inner product with every root. We assume that $\langle \lambda, \alpha \rangle < 0$ for $\alpha \in \Delta^+$. We recall from the structure theory that, in this case, C_0 is connected and hence abelian. In particular L is a character of C_0.

10.1. Proposition. *Under the above assumptions, the cohomology groups* $H^q(X, \mathscr{F})$ *vanish unless* $q = \operatorname{codim}_{\mathbb{C}} Q$, *in which case* $H^q(X, \mathscr{F})$ *is the minimal globalization of the standard Harish-Chandra module* $I(x, L)$.

Proof. With minor modifications, this is essentially the result of Schmid's thesis [32]. The restriction $\mathscr{L}$ of $\mathscr{F}$ to S is a sheaf of sections of a holomorphic G_0-homogeneous vector bundle defined on S, and $H^q(X, \mathscr{F}) \cong H^q_c(S, \mathscr{L})$ as G_0-modules. Let $\mathscr{L}^*$ denote the sheaf of holomorphic sections of the dual bundle, and Ω the sheaf of sections of top dimensional holomorphic forms on S. Under some additional assumptions on λ it is shown in [32] that $H^q(S, \mathscr{L}^* \otimes \Omega) = 0$, unless $q = \dim_{\mathbb{C}} Q$, in which case $H^q(S, \mathscr{L}^* \otimes \Omega)$ is a Fréchet globalization of the (discrete series) Harish-Chandra module $I(x, L)^*$ contragredient to $I(x, L)$. A careful examination of the argument in [32] shows that $H^q(S, \mathscr{L}^* \otimes \Omega)$ is actually the maximal globalization of $I(x, L)$. The above-mentioned additional condition on λ can be removed using, by now standard, tensoring techniques (cf. [42]). Serre's duality theorem [35, Sect. 3, Théorème 2] now implies that, for each q, $H^q(X, \mathscr{F}) \cong H^q_c(S, \mathscr{L})$ is the topological dual of $H^{n-q}(S, \mathscr{L}^* \otimes \Omega)$. The proposition follows: the continuous dual of the maximal globalization is the minimal globalization of the contragredient Harish-Chandra module.

For the purpose of certain inductive arguments, we need a version of Proposition 10.1 in a slightly more general situation. We drop the assumptions of connectedness and semisimplicity on G_0. Instead we require that G_0 is in the *Harish-Chandra class*, i.e.,

(a) G_0 is reductive;

(b) $\operatorname{Ad}(G_0) \subset \operatorname{Ad}(\mathfrak{g})$;

(c) G_0 has finite number of connected components;

(d) the analytic subgroup of G_0 corresponding to $[\mathfrak{g}_0, \mathfrak{g}_0]$ has finite center.

Each such G_0 contains a maximal compact subgroup K_0, which meets every connected component of G_0 [18, Sect. 3].

Note that G_0 may have a nontrivial vector group in its center. If this is the case, a Cartan subgroup of K_0 cannot be a Cartan subgroup of G_0. In what follows, the equal rank condition of Proposition 10.1 is replaced by the condition that the anisotropic component ${}^0G_0$ of G_0 has the same rank as K_0. Recall that ${}^0G_0$ is, by definition, the maximal subgroup of G_0 on which every continuous positive multiplicative character is trivial. The group G_0 decomposes as a direct product of the vector group factor of its center and ${}^0G_0$. Thus the center of ${}^0G_0$ is compact, and $K_0 \subset {}^0G_0$. In this case, as $\mathfrak{c}$, we choose a σ, τ-stable Cartan subalgebra of $\mathfrak{g}$ containing a

σ, τ-stable Cartan subalgebra of $\mathfrak{k}$ and, as x, we choose any point for which $\mathfrak{b}_x$ contains $\mathfrak{c}$. The Cartan subgroup C_0 of G_0 corresponding to $\mathfrak{c}$, which is also the stabilizer of x in G_0, is in general no longer connected or abelian.

Let K be the complexification of K_0. We denote by S (resp. Q) the G_0-orbit (resp. K-orbit) of x. As before, S is open, Q is compact, and K_0 acts transitively on Q. We emphasize that in general S and Q are not connected. We can define $I(x, L)$ and $\mathscr{F} = \mathscr{I}(x, L)$ as before. Here L is an irreducible $(\mathfrak{b}_x, C_0)$-module, with the action of $\mathfrak{b}_x$ determined by $\lambda_x + \rho_x$.

10.2. PROPOSITION. *If G_0 is a group in the Harish-Chandra class for which ${}^0G_0$ and K_0 have the same rank and if x is chosen as above, then the cohomology groups $H^q(X, \mathscr{F})$ vanish unless $q = \operatorname{codim}_{\mathbb{C}} Q$, in which case $H^q(X, \mathscr{F})$ is the minimal globalization of the standard Harish-Chandra module $I(x, L)$.*

Proof. Schmid's proof of 10.1 applies without change if G_0 is connected and reductive with compact center. If G_0 is connected and reductive with a nontrivial vector component in its center, we may simply apply this result to $H^q(X, \mathscr{F})$ and $I(x, L)$ regarded as ${}^0G_0$-modules. If G_0 is not connected, we first need to consider an intermediate group $(G_0)^\dagger = Z \cdot (G_0)^0$, where Z is the centralizer, in G_0, of its connected component of identity $(G_0)^0$. Since C_0 contains Z, the extension to $(G_0)^\dagger$ causes no problems. Finally, the extension from $(G_0)^\dagger$ to G_0 is just a finite induction (cf. [18, Sect. 27]). The corresponding extension on the level of standard modules has been carried out in the appendix of [19].

We now return to our usual conditions on G_0 (i.e., connected, semisimple, and finite center) but drop the equal rank assumption. We fix an arbitrary σ, τ-stable Cartan subalgebra $\mathfrak{c} \subset \mathfrak{g}$. We decompose $\mathfrak{c}$ into a direct sum $\mathfrak{t} \oplus \mathfrak{a}$ of 1 and (-1)-eigenspaces of τ. A root in $\Delta(\mathfrak{c})$ is called a $\mathfrak{t}$-*root* if its restriction to $\mathfrak{a}$ is zero. Let $\mathfrak{b}_x$ be a Borel subalgebra containing $\mathfrak{c}$. We say that $\mathfrak{b}_x$ is *of Langlands type*, if for every $\alpha_x \in \Delta_x^+$ which is not a $\mathfrak{t}$-root (i.e., either real or complex) we have $\sigma\alpha_x \in \Delta_x^+$. The direct sum of root spaces corresponding to such roots forms a σ-stable ideal $\mathfrak{u}$ of $\mathfrak{n}_x$. The normalizer $\mathfrak{p}$ of $\mathfrak{u}$ in $\mathfrak{g}$ is a σ-stable parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}_x$. We note that $\mathfrak{u}$ is the nilpotent radical of $\mathfrak{p}$. If $\mathfrak{m}$ is the Lie subalgebra generated by $\mathfrak{t}$ and $\mathfrak{t}$-root spaces, and $\mathfrak{l} = \mathfrak{m} \oplus \mathfrak{a}$, then $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ and $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{u}$ are the Levi and Langlands decompositions of $\mathfrak{p}$, respectively.

Let P_0 be the real parabolic subalgebra of G_0 corresponding to $\mathfrak{p}$, i.e., the normalizer of $\mathfrak{p}$ in G_0. On the level of groups, we have the decompositions: $P_0 = L_0 U_0 = M_0 A_0 U_0$, where L_0 is the centralizer of $\mathfrak{a}$ in G_0, U_0 is the unipotent radical of P_0, and A_0, M_0 are the split and anisotropic components of L_0. Also, $K_0 \cap M_0$ is a maximal compact subgroup of M_0 and T_0, the centralizer of $\mathfrak{t}$ in M_0, is a compact Cartan subgroup of M_0, which is contained in $K_0 \cap M_0$. Thus, P_0 is a cuspidal parabolic subalgebra of G_0.

Consider the (projective) variety Y consisting of the parabolic subalgebras of $\mathfrak{g}$ which are conjugate under G to $\mathfrak{p}$. Let y be the point of Y parametrizing $\mathfrak{p}$. Thus $\mathfrak{p} = \mathfrak{p}_y$. Let $\pi: X \to Y$ be the natural projection. The fiber $X_y = \pi^{-1}(y)$ can be regarded, in a natural way, as the flag variety of $\mathfrak{p}/\mathfrak{u} \cong \mathfrak{l}$. We want to describe the restriction of π to the G_0-orbit S and to the K-orbit Q of x. We have: $\pi(S) \cong G_0/P_0 \cong K_0/K_0 \cap L_0$. On the other hand, $S \cap X_y$ is the L_0-orbit of x, and the stabilizer of x in L_0 is $T_0 A_0$. Similarly, $\pi(Q) \cong K/J_y$, where J_y is reductive (its Lie algebra is $\mathfrak{k} \cap \mathfrak{l}$). It follows that $\pi(Q)$ is affine and open and dense in Y. Moreover, $\pi(S)$ is a real compact form of $\pi(Q)$, and therefore a compact Stein set. Finally, $Q \cap X_y$ is the $K_0 \cap L_0$-orbit of x.

As we have mentioned above, X_y can be regarded as the flag variety of $\mathfrak{l}$. Now, L_0 is a Lie group in the Harish-Chandra class with complexified Lie algebra $\mathfrak{l}$ (cf. [18, Sect. 4]), $L_0 \cap K_0$ is its maximal compact subgroup, and ${}^0L_0$ and ${}^0L_0 \cap K_0$ have equal rank. Proposition 10.2 is therefore applicable in this situation. To make the statement precise, we first note that the abstract Cartan subalgebras of $\mathfrak{g}$ and $\mathfrak{l}$ are naturally isomorphic; in what follows we do not distinguish between them. Next, we identify, as we may, the root system $\Delta(\mathfrak{c}, \mathfrak{l})$ of $\mathfrak{c}$ in $\mathfrak{l}$ with the set of $\mathfrak{t}$-roots in $\Delta(\mathfrak{c})$. We write $\rho_x = \rho_{x,\mathfrak{t}} + \rho_{x,\mathfrak{a}}$, where $\rho_{x,\mathfrak{t}}$ is one-half of the sum of positive $\mathfrak{t}$-roots. The fact that $\mathfrak{b}_x$ is of Langlands type implies that $\rho_{x,\mathfrak{t}}$ and $\rho_{x,\mathfrak{a}}$ vanish on $\mathfrak{a}$ and $\mathfrak{t}$, respectively.

Let L be an irreducible $(\mathfrak{b}_x, C_0)$-module such that its differential is of the form $\lambda_x + \rho_x$, where λ_x satisfies

$$\langle \lambda_x, \alpha_x \rangle < 0 \qquad \text{for every } \mathfrak{t}\text{-root} \quad \alpha_x \in \Delta_x^+. \tag{10.1}$$

In view of the above remark, (10.1) implies that $\langle \lambda_x + \rho_{x,\mathfrak{a}}, \alpha_x \rangle < 0$, for every $\mathfrak{t}$-root $\alpha_x \in \Delta_x^+$. This allows us to define the standard Harish-Chandra modules $I(Q \cap X_y, x, L)$, and standard sheaves $\mathscr{I}(S \cap X_y, x, L)$, in the context of $\mathfrak{l}$, L_0, and $L_0 \cap K_0$. Proposition 10.2 now asserts that:

10.3. Lemma. *$H^q(X_y, \mathscr{I}(S \cap X_y, x, L)) = 0$, unless $q = \dim X_y - \dim(Q \cap X_y)$, in which case $H^q(X_y, \mathscr{I}(S \cap X_y, x, L))$ is the minimal globalization of $I(Q \cap X_y, x, L)$.*

Let us now investigate the cohomology groups $H^q(X, \mathscr{F})$ of an induced $(\mathscr{D}_\lambda, G_0)$-module $\mathscr{F} = \mathscr{F}(x, L)$ under the conditions outlined above; that is, λ satisfies (10.1), $\mathfrak{b}_x$ is of Langlands type, and L is an irreducible $(\mathfrak{b}_x, C_0)$-

module. There exists a spectral sequence, namely the Leray spectral sequence of the projection $\pi\colon X \to Y$ such that

$$E_2^{p,q} = H^p(Y, R^q\pi_*(\mathscr{F})) \Rightarrow H^{p+q}(X, \mathscr{F}) \tag{10.2}$$

(cf. [7, Chap. IV, Sect. 6; 15, Chap. 3, Sect.7]). Since $D^q\pi_*(\mathscr{F})$ has a Hausdorff geometric fiber $V = H^q(X_y, \mathscr{I}(S \cap X_y, x, L))$ at y, it is obtained by analytic induction from V. Thus $R^q\pi_*(\mathscr{F}) \cong \mathscr{I}(y, V)$, where $\mathscr{I}(y, V)$ denotes the induced module (cf. Sect. 8). By Lemma 10.3, $V = 0$, unless $q = \dim X_y - \dim(Q \cap X_y)$, in which case it is the minimal globalization of $I(Q \cap X_y, x, L)$. Since $\dim X_y - \dim(Q \cap X_y) = c(S) = \dim X - \dim Q$, it follows that the spectral sequence (10.2) degenerates, and

$$H^q(X, \mathscr{F}) = H^{q-c(S)}(Y, \mathscr{I}(V)).$$

10.4. Lemma. *Let V be an analytic DNF L_0-module supported on $\pi(S)$. Then $H^p(Y, \mathscr{I}(y, V)) = 0$ for $p > 0$.*

Proof. Note that $H^p(\pi(S), \mathcal{O}) = 0$ for $p > 0$, since $\pi(S)$ is a compact Stein set.

Suppose now that V is a restriction to L_0 of an analytic G_0-module. Then $\mathscr{I}(y, V) \cong (\mathcal{O}_Y \hat{\otimes} V)|_{\pi(S)}$ in a canonical G_0-invariant way. We claim that, in this case, $H^p(Y, \mathscr{I}(V)) = H^p(\pi(S), \mathcal{O}_Y \hat{\otimes} V) = 0$ for $p > 0$. In fact, let $\mathscr{C}^*(\mathcal{O}_Y)$ be the Čech resolution of $\mathcal{O}_Y$. Then $\mathscr{C}^*(\mathcal{O}_Y) \hat{\otimes} V$ is a resolution of $\mathcal{O}_Y \hat{\otimes} V$ by a complex of fine, hence acyclic, DNF sheaves. Therefore $H^p(\pi(S), \mathcal{O}_Y \hat{\otimes} V) = H^p(\Gamma(\pi(S), \mathscr{C}^*(\mathcal{O}_Y) \hat{\otimes} V))$. Since $\Gamma(\pi(S), \mathcal{O}_Y) \to \Gamma(\pi(S), \mathscr{C}^*(\mathcal{O}_Y))$ is an exact complex of DNF spaces, so is $\Gamma(\pi(S), \mathcal{O}_Y) \hat{\otimes} V \to \Gamma(\pi(S), \mathscr{C}^*(\mathcal{O}_Y)) \hat{\otimes} V$. The operations of completed tensor product and global sections over compact sets commute for DNF sheaves (cf. Lemma 3.6), and we conclude that the complex $\Gamma(\pi(S), \mathcal{O}_Y \hat{\otimes} V) \to \Gamma(\pi(S), \mathscr{C}^*(\mathcal{O}_Y) \hat{\otimes} V)$ is exact. This proves the claim.

Now, let V be an arbitrary analytic DNF L_0-module. Then, via evaluation at y, V is a quotient of the analytic DNF G_0-module $F_0(V) = \Gamma(\pi(S), \mathscr{I}(y, V))$. Continuing, we get a left resolution $F^*(V) \to V$ of V by analytic DNF G_0-modules. By inducing at y, we get a resolution $\mathscr{I}(y, F^*(V)) \to \mathscr{I}(y, V)$ of $\mathscr{I}(y, V)$ by acyclic sheaves. Since the functor $\Gamma(\pi(S), -)$ has finite cohomological dimension, it follows that $\mathscr{I}(y, V)$ is acyclic.

Thus we have shown that, under assumption (10.1), $H^q(X, \mathscr{F}) = 0$, unless $q = c(S) = \operatorname{codim} Q$, in which case it is isomorphic, as a topological G_0-module, to $\Gamma(\pi(S), \mathscr{I}(y, U^\sim))$, where $U^\sim = H^{c(S)}(X_y, \mathscr{I}(S \cap X_y, x, L))$ is the minimal globalization of the Harish-Chandra module $U = \Gamma(X_y, \mathscr{I}^{\mathrm{alg}}(Q \cap X_y, x, L))$. In particular, $H^{c(S)}(X, \mathscr{F})$ is an analytic DNF G_0-module.

It is quite easy to identify $\Gamma(\pi(S), \mathscr{I}(U^{\sim}))$. View $U^{\sim}$ as a P_0-module, by letting the unipotent radical act trivially, and consider the space $\mathrm{Ind}(U^{\sim})$ consisting of real analytic functions $f: G_0 \to U^{\sim}$ which satisfy $f(gp) = p^{-1}f(g)$ for $g \in G_0$, $p \in P_0$. The group G_0 acts on $\mathrm{Ind}(U^{\sim})$ by left translations. Since $\pi(S) = G_0/P_0$, $\mathrm{Ind}(U^{\sim})$ is isomorphic, as a topological G_0-module, to $\Gamma(\pi(S), \mathscr{I}(y, U^{\sim}))$. The functor Ind is the "classical" (unnormalized) parabolic induction functor. It has several variants. For example, $U^{\sim}$ can be replaced by a Banach space globalization $U^{\wedge}$ of U, and "f real analytic" can be substituted by "f continuous." Topologize the resulting space $\mathrm{Ind}(U^{\wedge})$ by "sup on K_0" norm. In this way $\mathrm{Ind}(U^{\wedge})$ becomes a Banach space representation of G_0. The underlying Harish-Chandra module is independent of the globalization of U, and of a particular topological or functional analytic variant of Ind. In fact, it can be constructed intrinsically in terms of U—one such realization will be presented below—and is denoted here by $\mathrm{Ind}(U)$.

Now, $\mathrm{Ind}(U^{\sim})$ is contained in the space of analytic vectors in a Banach space globalization $\mathrm{Ind}(U^{\wedge})$ of $\mathrm{Ind}(U)$. Therefore, by Proposition 7.3, $\mathrm{Ind}(U^{\sim})$ is the minimal globalization of $\mathrm{Ind}(U)$. Consider the Harish-Chandra module $I(x, L) = \Gamma(X, \mathscr{I}^{\mathrm{alg}}(x, L))$. We have a string of $(\mathfrak{g}, K)$-invariant identifications: $I(x, L) \cong \Gamma(X, \mathscr{I}^{\mathrm{alg}}(x, L)) \cong \Gamma(\pi(Q), \pi_*(\mathscr{I}^{\mathrm{alg}}(x, L)|_Q))$. If we note that the geometric stalk of $\mathscr{I}^{\mathrm{alg}}(x, L)$ at y is U and that $\pi(Q) = K/J_y$, then we conclude that $I(x, L)$ is the space of J_y-invariants in $\mathcal{O}_K^{\mathrm{alg}} \otimes U$. Here $\mathcal{O}_K^{\mathrm{alg}}$ denotes the ring of regular functions on K, and J_y acts on $\mathcal{O}_K^{\mathrm{alg}} \otimes U$ by tensor product of the right translation representation on the first factor, and the natural action on the second. An application of the Peter–Weyl theorem shows that the space of J_y-invariants in $\mathcal{O}_K^{\mathrm{alg}} \otimes U$ is isomorphic to the Harish-Chandra module of $\mathrm{Ind}(U^{\sim})$. Consequently, $I(x, L) \cong \mathrm{Ind}(U)$. We collect the above results in the following proposition.

10.5. PROPOSITION. *Assume that $x \in X$ is of Langlands type and that L is an irreducible* $(\mathfrak{b}_x, C_0)$-module *satisfying condition* (10.1). *Then the cohomology groups $H^q(X, \mathscr{F})$ of the standard* $(\mathscr{D}_\lambda, G_0)$-module $\mathscr{F} = I(x, L)$ *vanish unless $q = c(S) = \dim X - \dim Q$. Moreover, $H^{c(S)}(X, \mathscr{F})$ is the minimal globalization of the Harish–Chandra module $I(x, L)$.*

We now impose the following condition on a regular $\lambda \in \mathfrak{b}^*$:

$$\mathrm{Re}\langle \lambda, \alpha \rangle \leqslant 0 \qquad \text{for} \quad \alpha \in \Delta^+. \tag{10.3}$$

Let $\mathscr{F} = \mathscr{I}(x, L)$ be an arbitrary standard $(\mathscr{D}_\lambda, G_0)$-module. Our goal is to prove an analogue of Proposition 10.5 in this situation.

Recall that the Borel subalgebra $\mathfrak{b}_x$ is determined by a pair $(\mathfrak{c}, \Delta_x^+)$, where $\mathfrak{c}$ is a σ, τ-stable Cartan subalgebra of $\mathfrak{g}$, and Δ_x^+ is a system of positive roots in $\Delta(\mathfrak{c})$. Recall also from the previous section the notions of various types of roots in $\Delta(\mathfrak{c})$ (i.e., complex, real, etc.), as well as the decomposition $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{a}$. It will be convenient to denote λ_x by another symbol, say γ.

10.6. Lemma. *If λ satisfies* (10.3), *there exists a sequence of positive root systems $\{\Sigma_k\}_{1 \leqslant k \leqslant m}$ of $\Delta(\mathfrak{c})$ with the following properties*:

(a) $\Sigma_1 = \Delta_x^+$;

(b) *For every $k \leqslant m-1$, Σ_{k+1} is obtained from Σ_k by reflection about a complex root $\beta_k \in \Sigma_k$, simple for Σ_k, with the property that $\sigma(\beta_k) \notin \Sigma_k$*;

(c) *Σ_m is of Langlands type. Moreover, $\langle \gamma, \beta \rangle < 0$ for every $\mathfrak{t}$-root $\beta \in \Sigma_m$, and $\operatorname{Re}\langle \gamma + \sigma\gamma, \beta \rangle \leqslant 0$ for every $\beta \in \Sigma_m$ which is real and complex.*

A proof of this lemma, in a different but equivalent formulation, is given in [20].

Denote by S_k and Q_k the G_0 and K-orbits of the point $x_k \in X$ determined by $\mathfrak{c}$ and Σ_k. Let $\alpha_k \in \Delta^+$ be the abstract simple root, whose specialization at x_k is β_k. By $\rho_k \in \mathfrak{c}^*$ we denote the specialization of ρ at x_k. We note that $\rho_k - \rho_1$ is a sum of roots and therefore lifts canonically to a one-dimensional C_0-module $\mathbb{C}(\rho_k - \rho_1)$. Finally, set $w(k) = s(\alpha_{k-1}) \cdots s(\alpha_1)$ for $k \geqslant 2$, and $w(1) = e$.

Define $\mathscr{F}_k = \mathscr{I}(x_k, L \otimes \mathbb{C}(\rho_k - \rho_1))$. This is a standard $(\mathscr{D}_{w(k)\lambda}, G_0)$-module supported on S_k. Similarly, $I_k = \Gamma(X, \mathscr{I}^{\mathrm{alg}}(x_k, L \otimes \mathbb{C}(\rho_k - \rho_1))$ is a $(U_{w(k)\lambda}, K)$-module. For $k = 1$ this is the standard Harish-Chandra module $I(x, L)$.

10.7. Lemma. *For each q, we have $H^q(X, \mathscr{F}_k) \cong H^{q-1}(X, \mathscr{F}_{k+1})$.*

Proof. By Proposition 9.10 the intertwining functor $\mathscr{T}_{(\alpha_k, w(k+1)\lambda)}$ associates $\mathscr{F}_k[1]$ to $\mathscr{F}_{k+1}$. We appeal now to Theorem 5.4 and Proposition 6.4 to conclude that $H^*(X, \mathscr{F}_k[1]) \cong H^*(X, \mathscr{F}_{k+1})$, which is equivalent to the statement of the lemma.

Now, by Lemma 10.6, x_m is of Langlands type, and condition (10.1) is satisfied. Thus we can apply Proposition 10.5 to conclude that $H^q(X, \mathscr{F}_m)$ vanishes, unless $q = c(S_m)$, in which case it is the minimal globalization of I_m. Now, it is shown in [20] that all I_k are isomorphic as Harish-Chandra modules. It is also a known fact about the structure of K-orbits on X that $\dim Q_{k+1} = \dim Q_k + 1$ (cf. [39, 20]). We can now deduce the following proposition.

10.8. Proposition. *If* (10.3) *is satisfied, $\mathscr{F} = \mathscr{I}(x, L)$ is a standard $(\mathscr{D}_\lambda, G_0)$-module, and S is the G_0-orbit of x, then the hypercohomology of*

$\mathscr{F}[c(S)]$ vanishes in degrees different from zero, and $H^0(X, \mathscr{F}[c(S)])$ is the minimal globalization of the standard module $I(x, L)$.

We have now all the ingredients to prove our main theorem.

10.9. THEOREM. *If $\lambda \in \mathfrak{h}^*$ is regular, then the functors $D\Delta$ and $D\Gamma$ establish an equivalence of the categories $D\mathfrak{M}(U_\lambda, G_0)$ and $D\mathfrak{M}(\mathscr{D}_\lambda, G_0)$.*

Proof. In view of Proposition 9.7 and the fact that intertwining functors commute with localization, it is enough to show the theorem for any particular element in the Weyl group orbit of λ. Thus we may assume that λ satisfies (10.3). Proposition 10.8 now asserts that $D\Gamma$ maps a generating set of the triangulated category $D\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ onto a generating set of the triangulated category $D\mathfrak{M}(U_\lambda, G_0)$. The proof follows now from Theorem 5.4, and the fact that $D\Gamma$ is a ∂-functor.

As an application of the above theorem we get an alternate characterization of minimal globalizations of Harish-Chandra modules, in terms of analyticity and finiteness of $\mathfrak{n}$-homology.

10.10. PROPOSITION. (a) *A minimal globalization of a Harish-Chandra module with infinitesimal character ϕ_λ has finite dimensional $\mathfrak{n}$-homology for every maximal nilpotent subalgebra $\mathfrak{n}$ of $\mathfrak{g}$.*

(b) *Conversely, if M_ω is a* DNF *analytic (U_λ, G_0)-module with finite dimensional $\mathfrak{n}$-homology for every maximal nilpotent subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ then M_ω is a minimal globalization of a Harish-Chandra module.*

Proof. Assume that M_ω is a minimal globalization of its Harish-Chandra module. Then M_ω is DNF and analytic. By Theorem 10.9 each localization $\Delta_p(M_\omega)$ is an analytic $(\mathscr{D}_\lambda, G_0)$-module with finite dimensional—hence Hausdorff—geometric fibers. It follows from Proposition 8.5 that $H_p(\mathfrak{n}, M_\omega)^{\lambda+\rho}$ is finite dimensional. Since we can replace λ by any of its Weyl group conjugates we see that all $\mathfrak{n}$-homologies are finite dimensional.

Now suppose that M_ω is a DNF analytic (U_λ, G_0)-module with finite dimensional $\mathfrak{n}$-homologies. We deduce from the proof of Proposition 8.5 that $\Delta_p(M_\omega)$ is an analytic $(\mathscr{D}_\lambda, G_0)$-module with finite dimensional geometric fibers. Thus, by Proposition 9.3, $D^p\Delta(M_\omega)$ is an object in $\mathfrak{M}(\mathscr{D}_\lambda, G_0)$ for each p. Consequently by our main theorem $M_\omega \cong D^0\Gamma(D\Delta(M_\omega))$ is an object in $\mathfrak{M}(U_\lambda, G_0)$. Thus the statement of the proposition follows from the lemma below.

10.11. LEMMA. *Suppose that a module M in $\mathfrak{M}(U_\lambda, G_0)$ lifts to a representation of G_0. Then the set of K-finite vectors in M is a Harish-Chandra module and M is its minimal globalization.*

Proof. A simple induction reduces the proof to the following statement: Let $0 \to M_1 \to M \to M_2 \to 0$ be an exact sequence of G_0-modules in $\mathfrak{M}(U_\lambda, G_0)$, where M_1 and M_2 are minimal globalizations. Then M is a minimal globalization. To prove this, we note that we get an exact sequence of the underlying spaces of K-finite vectors. For M_1 and M_2 these are Harish-Chandra modules and so it follows that this is true of M as well. By exactness of the minimal globalization functor (cf. Proposition 7.3) we get an exact sequence $0 \to M_1 \to M' \to M_2 \to 0$, where M' is the minimal globalization of the Harish-Chandra module of M. There is also a morphism $M' \to M$ which makes the diagram below commutative:

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] & M \arrow[r] & M_2 \arrow[r] & 0 \\ 0 \arrow[r] & M_1 \arrow[r] \arrow[u] & M' \arrow[r] \arrow[u] & M_2 \arrow[r] \arrow[u] & 0. \end{tikzcd}$$

Since the left and right vertical arrows in this diagram are identities the middle arrow is an algebraic isomorphism. By the open mapping theorem it must be a topological isomorphism. This concludes the proof of the lemma.

As we have mentioned in the introduction, there is a formal resemblance between the kinds of sheaves arising from localization of analytic G_0-modules and the sheaves on X which are constructible with respect to the stratification induced by the G_0-action. We now describe this connection.

Assume first that $\lambda = -\rho$. Then $\mathscr{D}_\lambda = \mathscr{D}_{-\rho}$ is nothing but the sheaf $\mathscr{D}$ of differential operators, wth holomorphic coefficients, on X. For a $\mathscr{D}$-module $\mathscr{M}$ we denote by $DR(\mathscr{M})$ the *de Rham complex* of $\mathscr{M}$:

$$0 \to \mathscr{M} \to \Omega^1 \otimes_{\mathcal{O}} \mathscr{M} \to \cdots \to \Omega^k \otimes_{\mathcal{O}} \mathscr{M} \to \cdots .$$

Here Ω^k denotes the sheaf of holomorphic differential forms on X of degree k. If $\mathscr{M}$ is an analytic $(\mathscr{D}, G_0)$-module of finite rank, the holomorphic Poincaré lemma implies that the cohomology of the above complex is non-vanishing only in degree zero, where it is the sheaf $\mathscr{M}^{\mathscr{D}}$ of flat local sections of $\mathscr{M}$. It is not difficult to check that $\mathscr{M}^{\mathscr{D}}$ is G_0-equivariant and constructible with respect to the stratification $\mathfrak{X}$ of X induced by the G_0-action. The canonical $\mathscr{D}$-module structure on $\mathcal{O}$ induces one on $\mathcal{O} \otimes \mathscr{M}^{\mathscr{D}}$ and the morphism $\mathcal{O} \otimes \mathscr{M}^{\mathscr{D}} \to \mathscr{M}$ is an isomorphism of $\mathscr{D}$-modules.

Assume now that $\mathscr{F}$ is a $\mathfrak{X}$-constructible sheaf of complex vector spaces. Then $\mathscr{F}$ is in a natural way a DNF sheaf. In fact, if K is compact and U open and $K \subset U$, then the restriction map $\Gamma(U, \mathscr{F}) \to \Gamma(K, \mathscr{F})$ has finite range (this follows from Wilder's "(P, Q) property," satisfied by constructible sheaves). Thus $\Gamma(K, \mathscr{F})$ is DNF, as an inductive limit of a countable system of finite dimensional spaces. For $K \subset K'$ compact, the map $\Gamma(U, \mathscr{F}) \to \Gamma(K, \mathscr{F})$ is clearly continuous.

Since $\mathscr{F}$ is a DNF sheaf, so is $\mathscr{M} = \mathcal{O} \otimes \mathscr{F}$, which is regarded as a $\mathscr{D}$-module, via the $\mathscr{D}$-module structure on $\mathcal{O}$. If $\mathscr{F}$ is G_0-equivariant, then $\mathscr{M}$ is an analytic $(\mathscr{D}, G_0)$-module of finite rank. We recover $\mathscr{F}$ from $\mathscr{M}$ as the subsheaf $\mathscr{M}^{\mathscr{D}}$ of flat local sections. Conversely, if we start with an analytic $(\mathscr{D}, G_0)$-module $\mathscr{M}$, then the isomorphism $\mathcal{O} \otimes \mathscr{M}^{\mathscr{D}} \to \mathscr{M}$ is topological. We have therefore shown the following proposition.

10.12. PROPOSITION. *The functor $\mathscr{M} \to \mathscr{M}^{\mathscr{D}}$ is an equivalence of the category of analytic $(\mathscr{D}, G_0)$-modules $\mathscr{M}$ of finite rank, and the category of $\mathfrak{X}$-constructible G_0-equivariant sheaves of complex vector spaces. The inverse functor is given by $\mathscr{F} \to \mathcal{O} \otimes \mathscr{F}$.*

If $\lambda + \rho$ is a weight of a finite dimensional representation of G—equivalently, if $\lambda + \rho$ determines a global line bundle $\mathscr{L}$ on X—then the above proposition can be generalized to the case of $\mathscr{D}_\lambda$. For a general twist, however, we are not aware of an analogue of the functor of taking flat local sections, which would allow us to identify analytic $(\mathscr{D}_\lambda, G_0)$-modules of finite rank with constructible sheaves. However, the above result suggests that the notion of an analytic $(\mathscr{D}_\lambda, G_0)$-module of finite rank is a good substitute for the notion of an $\mathfrak{X}$-constructible G_0-equivariant sheaf of complex vector spaces—the former notion makes sense for arbitrary λ and yields a category equivalent to that determined by the latter notion in the case $\lambda = -\rho$.

APPENDIX A: TOPOLOGICAL CONSIDERATIONS

For a great number of reasons it is important that most of the topological vector spaces we deal with in this paper belong to the category of strong duals of nuclear Fréchet spaces. We shall call nuclear Fréchet spaces NF spaces and strong duals of nuclear Fréchet spaces DNF spaces. In this section we gather the basic facts, mostly well known, about these spaces that we shall use throughout the paper.

A.1. The category of NF spaces is closed under passing to closed subspaces, separated quotients, countable projective limits, and completed tensor products [31, III, 7.4, 7.5].

A.2. Each NF space is Ptak [31, IV, 8.0], barreled [31, II, 7.1], reflexive, and in fact, Montel [31, IV, 5.6 and III, 7.2].

A.3. By duality, the category of DNF spaces is also closed under passing to closed subspaces and separated quotients as well as countable separated inductive limits and completed tensor products.

A.4. Each DNF space is reflexive and, hence, barreled [31, IV, 5.6] and Ptak [31, IV, 8.0] as well as nuclear [31, IV, 9.6].

We denote the completed projective tensor product of two topological vector spaces E and F by $E \hat{\otimes} F$.

A.5. If E is an NF space (resp. a DNF space), then $F \to E \hat{\otimes} F$ is an exact functor from NF spaces to NF spaces (resp. from DNF spaces to DNF spaces). Furthermore, there is a natural isomorphism $E \hat{\otimes} F \to L(E', F)$, where $L(E', F)$ is the space of continuous linear transformations from the strong dual E' of E to F [31, IV, 9.4, Cor. 1 and Cor. 2].

A topological homomorphism between two topological vector spaces is a linear map which is continuous and is an open map onto its image. A quotient map is, by definition of the topology on the quotient, a topological homomorphism. Furthermore, any topological homomorphism is the composition of a quotient map and an inclusion. We need a number of results telling us when we can be sure a map is a topological homomorphism.

The most general form of the open mapping theorem tells us that a surjection from a Ptak space onto a barreled space is a topological homomorphism [31, IV, 8.3, Cor. 1]. Similarly, every map from a barreled space to a Ptak space which has a closed graph is continuous. From the above properties of NF and DNF spaces we have:

A.6. Every continuous linear map between two NF or two DNF spaces is a topological homomorphism if it has closed range.

A.7. Every linear map between two NF or two DNF spaces is continuous if it has a closed graph.

A.8. PROPOSITION. *Let $\alpha: E \to F$ be a continuous map of complexes of NF spaces or DNF spaces. If for some p, the induced map on cohomology, $\alpha^*: H^p(E) \to H^p(F)$, is surjective, then it is also a topological homomorphism.*

Proof. The hypothesis means that the map

$$\alpha \oplus \partial: Z^p(E) \oplus F^{p-1} \to Z^p(F)$$

is surjective. Since $Z^p(F)$ is closed in F^p, the open mapping theorem applies and we conclude that $\alpha \oplus \partial$ is open. If we compose this with the quotient map $Z^p(F) \to H^p(F)$, then we have a continuous open map $\beta: Z^p(E) \oplus F^{p-1} \to H^p(F)$. Since the kernel of β contains $B^p(E) \oplus F^{p-1}$, β factors as a quotient map followed by $\alpha^*: H^p(E^*) \to H^p(F^*)$ which is therefore also an open map.

A.9. COROLLARY [29, Lemma 1 bis]. *If a map between complexes of DNF spaces is a quasi-isomorphism in the algebraic sense, then it is also a quasi-isomorphism in the sense of complexes of topological vector spaces—that is, the induced maps on cohomology will be topological as well as algebraic isomorphisms.*

Recall the definition of a distinguished triangle [4, Chap. 1; 38].

A.10. PROPOSITION. *If $E \to F \to G$ is a distinguished triangle of complexes of NF (resp. DNF) spaces and if $H^p(G)$ is Hausdorff, then $H^p(E) \to H^p(F)$ is a topological homomorphism.*

Proof. By the hypothesis that $E \to F \to G$ is a distinguished triangle and the corollary, we may assume without loss of generality that G is the cone $C(\alpha)$ over the map $\alpha: E \to F$. Recall that $C(\alpha)$ is defined so that $C^p(\alpha) = E^{p+1} \oplus F^p$ with boundary map given by $f \oplus g \to \partial f \oplus (\alpha f - \partial g)$: $E^p \oplus F^{p-1} \to E^{p+1} \oplus F^p$. Thus, the hypothesis that $H^p(G)$ is Hausdorff means that this map has closed range in $E^{p+1} \oplus F^p$. Since $\{0\} \oplus F^p$ is also closed in $E^{p+1} \oplus F^p$ and the intersection of closed subspaces is closed, we conclude that the map $f \oplus g \to \alpha f - \partial g: Z^p(E) \oplus F^{p-1} \to Z^p(F)$ also has closed range and, hence, is a topological homomorphism. Now the range of this map clearly contains $B^p(F)$ and so it remains a topological homomorphism if we compose it with the quotient $Z^p(F) \to H^p(F)$. Thus, we have a topological homomorphism $Z^p(E) \oplus F^{p-1} \to H^p(F)$. This clearly factors through $\alpha^*: H^p(E) \to H^p(F)$ and the latter map has the same range. We conclude that α^* is also a topological homomorphism.

A.11. COROLLARY. *Let $0 \to E \to F \to G \to 0$ be a short exact sequence of complexes of NF (resp. DNF) spaces. Let $A \to B \to C$ be any three consecutive terms in the associated long exact sequence. If C is Hausdorff, then $A \to B$ is a topological homomorphism. If, in addition, $A \to B$ has closed kernel, then B is Hausdorff.*

Proof. The first statement follows from the preceding proposition, the fact that any short exact sequence of complexes gives rise to a distinguished triangle, and the fact that triangles can be turned.

The second statement is proved as follows: If $A \to B$ has closed kernel and is a topological homomorphism, then the image of the complement of the kernel—i.e., the complement of zero in the image of $A \to B$—is open.

Hence, zero is closed in the image of $A \to B$, which, in turn, is closed because it is the kernel of a map to a Hausdorff space C. Thus, zero is closed in B.

APPENDIX B: Calculations on $\mathfrak{sl}_2(\mathbb{C})$

In this section we specialize to the case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $X = P^1$. Let $\mathscr{W}$ be an analytic $\mathscr{D}_\lambda$-module on X. For $a \in X$, we wish to compute the $\mathfrak{n}_x$-homology of the stalk $\mathscr{W}_a$ of this sheaf considered as a $\mathfrak{g}$-module. We use the results of such a calculation in Section 9. To this end, fix two different points a and b in X. This determines a Cartan subalgebra $\mathfrak{c} = \mathfrak{b}_a \cap \mathfrak{b}_b$ of $\mathfrak{g}$, for which $\mathfrak{n}_a$ and $\mathfrak{n}_b$ are the root spaces. We choose a basis e_a, e_b, h of $\mathfrak{g}$ such that $e_a \in \mathfrak{n}_a$, $e_b \in \mathfrak{n}_b$, $[h, e_a] = 2e_a$, $[h, e_b] = -2e_a$, and $h = [e_a, e_b]$. The positive root α on the abstract Cartan subalgebra $\mathfrak{h}$ specializes to roots α_a and α_b on $\mathfrak{c}$ corresponding to $\mathfrak{n}_a$ and $\mathfrak{n}_b$. With our choice of basis, $\alpha_a(h) = 2$ and $\alpha_b(h) = -2$. Similarly, any weight $\lambda \in \mathfrak{h}$ specializes to weights λ_a and λ_b on $\mathfrak{c}$ with $\lambda_a = -\lambda_b$.

The map $t \to \exp(te_b)a$ identifies $\mathbb{C}$ with the N_b orbit of a and $P^1 = \mathbb{C} \cup \{\infty\}$ with X. The points 0 and ∞ are sent to a and b, respectively. This same map gives us a trivialization of $\mathscr{W}_a$ as follows: Since $\mathscr{W}$ is analytic, the stalk $\mathscr{W}_a$ is induced from an analytic $\mathfrak{b}_a$-module L. That is, as a $\mathfrak{g}$-module, $\mathscr{W}_a$ can be realized as the space of germs at the identity in G of holomorphic L-valued functions on G which transform appropriately under the right action of $\mathfrak{c}$ (cf. Sect. 8). Now, the map $f \to \{t \to f(\exp(te_-))\}$ determines a continuous linear isomorphism of $\mathscr{W}_a$ to $\mathscr{O}_0 \otimes L$ which sends the e_b flat sections of $\mathscr{W}_a$ to the constant L-valued functions.

If we use the above map to transfer the action of $\mathfrak{g}$ on $\mathscr{W}_a$ to an action on $\mathscr{O}_0 \otimes L$, then clearly e_b will act as the operator $-d/dt$ acting in the $\mathscr{O}_0$-factor. Then the actions of e_a and h are completely determined by the Lie algebra relations in $\mathfrak{sl}_2(\mathbb{C})$ and the fact that they must act as first order holomorphic differential operators which at $\{0\}$ have the scalar values 0 and $m = \lambda_\alpha(h) + \rho_\alpha(h) = \lambda_\alpha(h) + 1$, respectively. The result is that, under the above identification, e_a, e_b, h act on $\mathscr{O}_0 \otimes L$ by acting on the first factor as the differential operators:

$$e_a = t^2 d/dt + mt, \qquad e_b = -d/dt, \qquad h = 2t d/dt + m.$$

We may now proceed with our calculation of $\mathfrak{n}_x$-homology. The $\mathfrak{h}$-modules $H_p(\mathfrak{n}_x, \mathscr{W}_a)$ for $p = 0, 1$ are isomorphic to the cokernel and kernel, respectively, of

$$v \otimes f \to vf \colon \mathfrak{n}_x \otimes \mathscr{W}_a \to \mathscr{W}_a.$$

If $x \neq a$ then we may choose $b = x$ in the above trivialization. Then $\mathfrak{n}_x$ is spanned by e_b and α_b is the positive root on $\mathfrak{h}$ determined by $\mathfrak{b}_x$. Thus, $H_0(\mathfrak{n}_x, \mathscr{W}_a) = 0$ and $H_1(\mathfrak{n}_x, \mathscr{W}_a)$ is $\mathfrak{n}_x \otimes L$ with h acting as

$$m - 2 = \lambda_a(h) - 1 = -\lambda_b(h) - 1 = s(\alpha)\, \lambda_b(h) + \rho_b(h);$$

thus, as an $\mathfrak{h}_x$-module it has weight $s(\alpha)\lambda_x + \rho_x$.

Now, let $x = a$ and choose b to be an arbitrary point of X not equal to x. The operator $td/dt + m$ is diagonalizable with the monomials t^n, for n a nonnegative integer, as eigenvectors. Thus, if $m = 2\lambda_\alpha(h) + 1$ is not a non-positive integer, then $td/dt + m$ is bijective on $\mathcal{O}_0$ and $e_a = t(td/dt + m)$ has (0) as kernel and $\mathcal{O}_0/t\mathcal{O}_0$ as cokernel. Hence, in this case, $H_1(\mathfrak{n}_x, \mathscr{W}_a)$ vanishes and $H_0(\mathfrak{n}_x, \mathscr{W}_a)$ is a copy of L with h acting as $m = \lambda_\alpha(h) + \rho_\alpha(h) = \lambda_x(h) + \rho_x(h)$. Thus, as an $\mathfrak{h}_x$-module, $H_0(\mathfrak{n}_x, \mathscr{W}_a)$ has weight $\lambda_x + \rho_x$.

If m is a nonpositive integer, then $\mathcal{O}_0$ decomposes as the direct sum of the zero eigenspace $\{\mathbb{C}t^{-m}\}$ for $td/dt + m$ and the span of the nonzero eigenspaces. Hence, e_a has kernel $\{\mathbb{C}t^{-m}\}$ and cokernel $(\mathcal{O}_0/t\mathcal{O}_0) \oplus (\mathcal{O}_0/T_{1-m})$, where T_{1-m} is the closed span in $\mathcal{O}_0$ of the monomials other than t^{1-m}. In this case, $H_1(\mathfrak{n}_x, \mathscr{W}_a)$ is a copy of L with h acting as $-m + 2 = -\lambda_\alpha(h) + \rho_\alpha(h) = s(\alpha_x)\, \lambda_x(h) + \rho_x(h)$, so that it has weight $s(\alpha_x)\, \lambda_x + \rho_x$ as $\mathfrak{b}_x$-module. On the other hand, $H_0(\mathfrak{n}_x, \mathscr{W}_a)$ is a direct sum of two copies of L, on one of which h acts as $-m + 2$ and on the others as m. Thus, $H_0(\mathfrak{n}_x, \mathscr{W}_a)$ is a direct sum of $\mathfrak{h}_x$-modules of weight $s(\alpha_x)\, \lambda_x + \rho_x$ and of weight $\lambda_x + \rho_x$, each of which has dimension $\dim(L)$. We conclude:

B.1. Proposition. *Suppose $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, $\mathscr{W}$ is an analytic $\mathscr{D}_\lambda$-module on X with finite geometric fibers, and $a \in X$.*

(a) *If λ is not a negative integral multiple of α and $\nu = s(\alpha_x)\, \lambda_x + \rho_x$, then $H_p(\mathfrak{n}_x, \mathscr{W}_a)^\nu$ is nonvanishing only for $x \neq a$ and $p = 1$, in which case it is isomorphic to $\mathfrak{n}_x \otimes L$;*

(b) *if λ is a negative integral multiple of α and $\nu = s(\alpha_x)\, \lambda_x + \rho_x$, then $H_p(\mathfrak{n}_x, \mathscr{W}_a)^\nu$ is nonvanishing but finite dimensional in both degrees $p = 0$ and $p = 1$;*

(c) *if $\nu = \lambda_x + \rho_x$, then $H_p(\mathfrak{n}_x, \mathscr{W}_a)^\nu$ is nonvanishing only for $x = a$ and $p = 0$, in which case it is isomorphic to $\mathfrak{n}_x \otimes L$.*

Note that part (c) illustrates our vanishing theorem of Section 4, while parts (a) and (b) give what is needed to complete the arguments in Section 9.

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